MATHEMATICS

FOR

MODERN LIFE

CURRICULUM

EPH P. McCORMACK

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MATHEMATICS
FOR
MODERN LIFE

BY

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"Plane Geometry", "Solid Geometry", etc.

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PREFACE

Years ago it was generally assumed that there was a complete transfer of training from school subjects to the problems of life, and little or no attempt was made in the schools to aid this transfer. Then the pendulum swung to the opposite extreme and a prominent psychologist startled the educational world by asserting that there was very little transfer of training. Now, however, we have gotten away from both extremes, and it is reasonably well established that the teacher can get as much transfer of training as he tries to get, but that he must teach his subject in such a way that it will transfer. He must consciously seek this transfer, and the best way to accomplish this is to make as many contacts with life situations as the nature of the subject-matter will permit.

The search for these contacts with life situations should not be left for the teacher who has neither the time nor the opportunity for such investigation. This is clearly the duty of the author. The modern textbook should be so written that (1) it interests the student, (2) it furnishes him with those contacts with life that will enable him to carry over his mathematical training into his later life, and (3) it develops in him those qualities and ideals which will better fit him for the life he is to live. It was with such objectives in mind that this book was written.

This book aims to interest the student. Its approach is natural. He is led from the arithmetic he already knows to the techniques of high-school mathematics by simple steps in which the new is constantly tied up with the old. Many topics begin with a review of his arithmetic, and the connection of the new work with the old is constantly emphasized. The sequence of topics is such as to create a need for the new topic before the study of it is begun. The explanation of the new topic is simple and clear and expressed in words that he understands. The book is written to the pupil. Nothing kills his interest more

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quickly than meeting difficulties that he cannot overcome, and nothing arouses his interest more enthusiastically than the knowledge that he is learning how to do something that he wants to do. Any subject is interesting to the student if it shows him how to solve the problems that he wants to solve. In this book he is shown from the very start the purpose of algebraic symbolism and its advantage in solving his problems over the clumsier methods he has previously used. He is led through a great wealth of practical problems which he either has already wanted to solve or knows that someone needs to solve as part of his daily life. And all this time the problems are kept within his ability so that he need not meet the discouragement of failure.

This book furnishes the student with those contacts with life that will enable him to carry over his mathematical training into his later life. Every topic that he studies is immediately applied to the problems of life. And great stress is laid on those problems that interest him now. For the boy, problems for the Boy Scout, finding heights and distances, baseball, radio, chemistry, aviation, building bridges, and all such problems that boys want to solve. For the girl, problems in the home, in cooking and sewing, in sports, problems for the Girl Scout, and all such problems that girls want to solve. In addition, both are shown that mathematics is the foundation of nearly every subject of study. They apply their new tools to the problems of the professional man, the physician, the chemist, the astronomer, the pharmacist, the physicist, the civil and electric engineer, to the problems of the business man, the banker, the accountant, the economist, the merchant, the broker, to the problems of the artist, the musician, the painter, to the problems of the farmer, and to a wide variety of the problems of everyday life. Whenever the topic furnishes a shortcut to the processes of arithmetic, the student is immediately given exercises in which this shortcut is applied, and it is not left to him to attempt to use in later life what he has never been taught to use. His mathematics will transfer to life situations because he is taught to transfer it to them and is trained in applying it. We all believe that mathematics is the very foundation of our modern civilization. This the book attempts to bring out. If we tell the student that nearly everything in life is based on mathematics, he may believe us, but he can never realize it as he will if he himself applies it to nearly every subject.

This book attempts to develop in the student those qualities and ideals which will better fit him for the life he is to live. In addition to those problems that bring out the advantages of right living as opposed to carelessness, disobedience and crime, such as statistics on accidents, on punishment of crime, on cost and value of education, etc., and to those problems that tend to make the student a better citizen because he knows how to apply the advantages of mathematics to the problems of his life, the objectives that mathematics is best fitted to develop are constantly kept in mind. Neatness and orderly thinking are demanded, for example, by the arrangement of the analysis of a problem; accuracy and confidence in his conclusions and results are developed by an insistence on checking, training in analytic thinking by presenting problems about which he will want to think even outside of his school requirements and whose solution will mean something to him; training to work up to the level of his ability by problems graded on three levels so that the teacher can select those best fitted to the individual needs of the pupil or the class, habits of reflective thinking by thought problems and by the difficult exercises at the end of each chapter, and so on.

The usual breach between cultural and vocational mathematics is avoided in this text by the expedient of using the problems of practical mathematics as the material for the processes of cultural mathematics. There is no good reason why the useless should be more cultural than the useful. Although this book is not a commercial arithmetic, it applies the principles of mathematics to so many of the problems of business life that the pupil has a better business foundation than he is likely to obtain from a purely business arithmetic. It is not a vocational text, but it uses such a variety of problems from the various vocations that it gives the student the training he needs as a foundation for any vocational pursuit. And at the same time it presents him with that mathematics most needed as a foundation if he continues his study of the subject as preparation for college. The best preparation for college is preparation for life. In fact, it goes further in this direction than is usual as, for example, in the treatment and use of logarithms which are customarily relegated to a second course in algebra.

Some of the features of this book are:

1. The development approach to new topics. This resembles the Investigation Problem of the author's geometries. Before the student is shown how to solve a new problem, he is asked a series of questions that should enable him to work out his own solution through his own activity.

2. The systematic grouping of related topics, as for example, the formula, graph, table, variation and dependence, or exponents, logarithms and slide rule.

3. Methods of thinking emphasized. In the demonstrative geometry, the object is training in fundamental methods of thinking rather than arriving at a particular theorem.

4. Emphasis on those basic concepts, the graph, the formula, the language of mathematics, the equation, the construction and solution of problems, relationship and dependence, ratio and proportion, generalizing and the general solution, and their application to the real problems of life.

5. Exercises graded on three levels of ability, and even graded within each level so that the teacher can easily select what is best fitted for his class.

6. Explanations that the pupil can understand. The book is written to the pupil and for the pupil, and the wording is kept down to his level.

7. Culture through the useful. Each mathematical principle is applied to the problems of modern life. Business problems and other problems that someone in his daily life needs to solve make mathematics real to the pupil and arouse his interest.

8. Geometry introduced as material for algebraic problems. The principal concepts of geometry, congruence, parallelism, similarity, the right triangle, areas, indirect measurement and numerical trigonometry, are all made familiar to the student through use.

9. The arrangement of the text is such that the material most likely to function in the life of the student is placed in the earlier chapters, with the more technical material postponed to later

chapters for those classes that have time for a more extended course.

10. A suitable treatment of symmetry.

11. A new method of introducing logarithms that brings them down to the level of the immature student, and that makes him enjoy using them.

12. The slide rule as an application of logarithms.

13. New-type tests and reviews with each chapter. These tests are graded from the extremely simple exercise at the beginning to the rather difficult final exercise.

14. This text covers all topics required by the syllabi in elementary algebra of the different states, and it meets completely the recommendations of the Joint Committee of the Mathematical Association of America and the National Council of Teachers of Mathematics.[1]

The author wishes to acknowledge his indebtedness to Emory F. White, Mary Doran, and J. Hood Branson for valuable suggestions, and to Mrs. Edna S. Whare, of Mont Pleasant High School, Schenectady, N. Y., and Paul C. Roundy, of Western Reserve Academy, Hudson, Ohio, for reading the manuscript.

J. P. McC.

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[1] *The Place of Mathematics in Secondary Education*, Report of the Joint Committee of the Mathematical Association of America and the National Council of Teachers of Mathematics, *Fifteenth Yearbook* of the National Council of Teachers of Mathematics (New York, Bureau of Publications, Teachers College, Columbia University, 1940).

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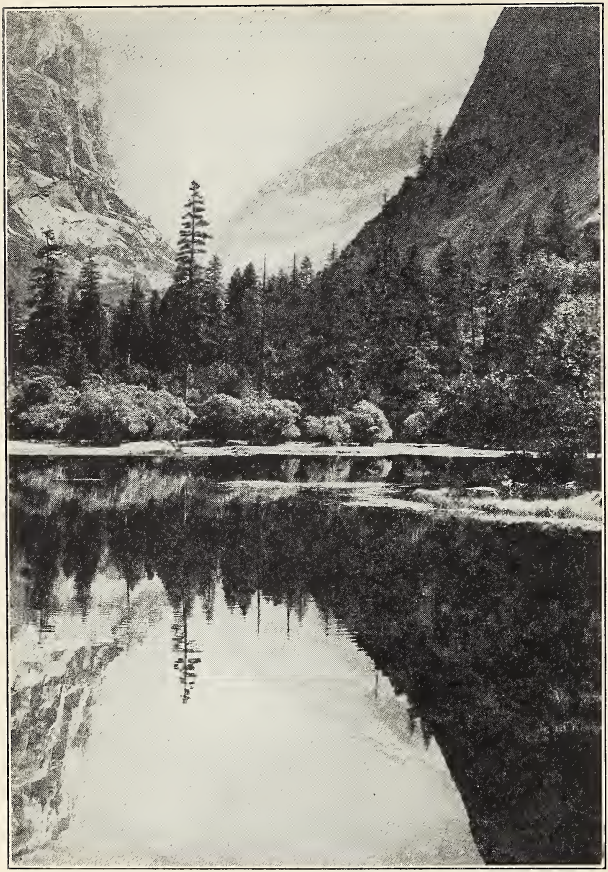


Photo from R. I. Nesmith and Associates.

MATHEMATICS IN NATURE

Notice how nature uses symmetry to add to the beauty of this scene. Can you find the axis of symmetry in this picture? Man, too, makes use of symmetry in fashioning our furniture, our dishes and silverware, and our automobiles.

Courtesy of the N.Y. Port Authority.

MATHEMATICS CARVED IN STEEL

Not many years ago the engineers who designed and constructed this bridge were studying the mathematics that you are now taking. Without mathematics, bridge construction would be impossible.

Introduction

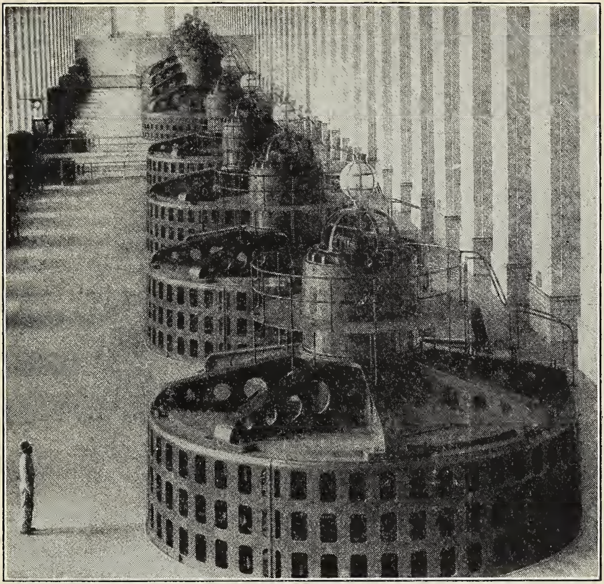
MATHEMATICS IN OUR WORLD TODAY

PROBABLY you who are about to begin the study of this book have little idea of the part mathematics plays in your life. You arise at a certain hour in the morning because your clock tells you the time. You know, of course, that your clock gains or loses time and that some one must correct it. But do you know that the astronomer, watching the stars through the night, uses a great deal of mathematics in doing this for you? And the man who designed your clock also used mathematics.

You may ride to school in an automobile, a street-car, or a bus. But if you went to the office of the company that manufactures any of these, you would find many engineers using mathematics in their work of designing a better car for you to ride in. You may cross a bridge or pass a tall building on your way to school. But did you ever study the great number of parts of a bridge, a bewildering maze of steel, pieces of different sizes, of different strengths, and in different positions? Some one had to determine the strength and size of each of those pieces so that it would stand the load that it must carry and be safe for you to cross. Do you know that every one of those pieces had been ordered and manufactured to the correct size before the construction of that bridge was begun? Some engineer had to know his mathematics to solve that problem.

Before your school or your home could be lighted by electricity or heated by steam, engineers had to use mathematics. Have you ever been in an electric power station? There you would see complicated machines, dynamos, with great coils of wire turning to furnish the electricity. Those coils were not put there by guess work. The great electric companies employ a large staff of engineers to design those machines, and every part is worked out by mathematics before the construction is begun. Also when you turn the dial of your telephone, did it ever occur to you that some one must have known a lot of mathematics to have worked out a machine that will get any telephone in your city or town for you by a few simple turns of the dial? Even the clothing you wear and the food you eat required machines in their manufacture, and these were designed by engineers who had spent years studying mathematics.

Your father pays taxes, either directly or indirectly, to support your town, your state, and your nation. Do you know how tax rates are determined and how your town's finances are worked out? Accountants do this. With the help of mathematics they determine the needs of your community and the distribution of the taxes. And the great banking system of your state requires men with technical knowledge based on mathematics. Our large stores too have a staff of statisticians so that they can



Courtesy of Tennessee Valley Authority.

AN ELECTRIC POWER STATION

Every part of these huge dynamos that generate electric power at the Wilson Dam was carefully checked by engineers. They used formulas, some of which you will use in this course.

buy more economically. You need mathematics even to read intelligently one of the better class of newspapers, for you must be able to understand and interpret graphs.

You are living in a scientific age. Science is revolutionizing your lives. You no doubt read of the great things being done by physicists and chemists. These men and women discover and invent new substances and new processes for your comfort and convenience that your ancestors never knew. Every year the world is turning more and more to science. And all science is built on a

Photograph by Charles Phelps Cushing.

FOR WORK AND FOR PLAY,
CREATIONS OF MATHEMATICS

foundation of mathematics. In fact the world of today is so dependent on mathematics that without it a large part of our modern civilization could not go on. And the world of tomorrow will be still more dependent on it.

This book can, of course, give you only the beginning of this wonderful science, but if you turn its pages and glance at the topics there, you will find that mathematics is useful in a surprisingly large number of the affairs of ordinary life such as cooking, art, music, and business. It will simplify for you many processes that you can do or have done by harder and more complicated methods. It will also enable you to solve many problems that you cannot solve at all without it.

But suppose you do not need mathematics in any of these lines. Of what use then is it to you? Mathematics trains you in correct reasoning and that is what the world needs most today. Suppose you never use your mathematics, suppose you forget the formulas and equations that you learn here, still if it helps to make you think more clearly and enables you better to reason out the ordinary problems of life, it will have given you something that will be of value to you long after the mathematics itself is forgotten.

Chapter 1

STATISTICAL GRAPHS

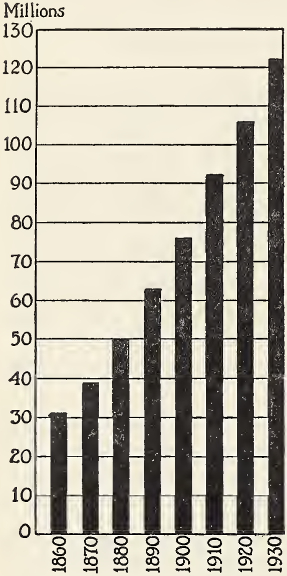
Why we study graphs. If someone talked to you for a long time describing the appearance of a man, you would not have a good idea of how he looked until you glanced at his photograph. To a lesser degree the same thing is true of statistics. Most people, in reading a scientific article or a daily newspaper, either skip the columns of figures altogether or at most get only a hazy idea of their relationships. But when these figures are pictured for them, they quickly notice many facts that they could have found in the figures only after considerable study. A picture showing these relationships of statistics is called a ***graph.***

Graphs are used for a great variety of purposes and are becoming more and more important every year. The better class of daily newspapers carry graphs of business conditions. Magazines and books of reference contain them; nurses use them to record a patient's condition; and every large business today has a statistical department in which a large force of clerks makes graphs to show how business is changing and how the corporation's money is being spent.

THE BAR GRAPH

Perhaps the simplest kind of graph is the bar graph such as is shown here. This graph appeared in a daily newspaper. Can you understand it? It shows the population of the United States for each census since 1860. The figures on the left tell the number of millions of people. For example, 60 means 60,000,000. The figures at the bottom tell the year in which the census was taken.

1. In 1890, the population was about 63,000,000. How does the graph show this?



Population of the United States.

2. What was the population in (*a*) 1880? (*b*) 1930? (*c*) 1910?

3. At what census did the population pass 100,000,000?

4. Has the population increased every decade? How does the graph tell you?

5. How many times as large was the population in 1930 as in 1860?

6. When was the population about half as large as it was in 1930?

7. How much did the population increase from 1920 to 1930?

8. Was the population increasing faster or slower in 1930 than in 1870?

What can we learn about the bar graph? 1. Are the bars all the same width?

2. Are the spaces between bars the same width?

3. Are all bars the same length? On what does the length of the bars depend?

4. Notice the numbers at the left. What number is at the bottom? Do you think that this is important?

5. Does the same difference in space always represent the same difference in millions on this graph?

6. If the largest population had been 240 million instead of 120 million, how many million should each interval represent if we wanted to obtain a graph of about the same size as this one?

7. Do the numbers at the bottom increase as we go toward the left or toward the right?

8. Do the numbers at the left increase as we follow them toward the top or toward the bottom?

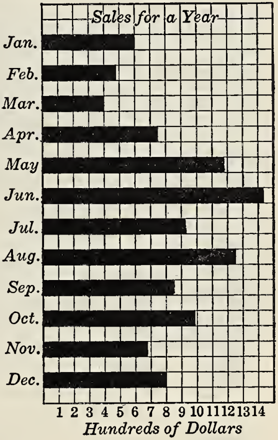
How to make a bar graph. 1. Draw a vertical line near the left side and a horizontal line near the bottom of the sheet. These are your *axes.*

2. Decide on a unit of such a size that all of your numbers will fit on the graph, and that the graph will be about as large as the space will allow.

3. Make all bars the same width and the spaces between them the same width.

4. Write the numbers at the left, beginning with 0 at the bottom and increasing as you go toward the top. Numbers differing by the same amount should always be the same number of spaces apart.

5. The numbers or names along the bottom should be equally spaced, beginning at the left.



The bar graph may be drawn with the bars running horizontally instead of vertically. In this case the time might be written along the vertical axis. This is the only kind of graph in which the time is so placed.

This graph shows the sales of a business house for a year.

1. In what month were the sales the greatest? The smallest?
2. What was the approximate amount of sales in April? In August?
3. In what month were the sales nearest $1000? $800?
4. By what amount did the sales in May exceed those in April?
5. In what month were the sales about twice those in March?

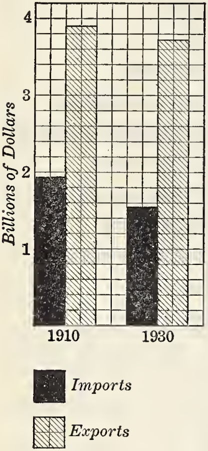
6. Were the sales greater in summer or in winter?

7. In which month were the sales the greater (*a*) January or November? (*b*) April or September? (*c*) May or August?

8. Was there as great an increase in sales from May to June as there was from April to May?

The double bar. Sometimes we wish to compare two kinds of data by putting two different graphs together. In this case the two sets of bars are shaded differently.

This graph compares the exports and imports of the United States for the years 1910 and 1930 for certain products.



Exercises

Make a bar graph illustrating the data in each of these exercises.

1. The casualties in the World War in millions were: Russia, 9; France, 6; British Empire, 3; Germany, 7; Italy, 2; Austria-Hungary, 7; and Turkey, 1.

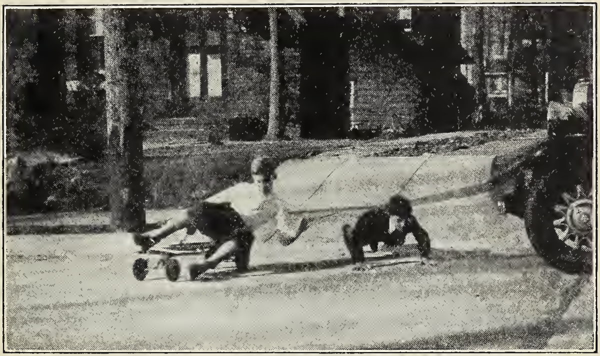
2. The highest peak on each continent in thousands of feet is: North America, 20; South America, 23; Europe, 18; Asia, 29; Africa, 20; Australia, 7; and Antarctica, 15.

3. The population (1930) of New York was 6,900,000; of Chicago, 3,400,000; of Philadelphia, 1,900,000; of Detroit, 1,600,000; and of Los Angeles, 1,200,000. (Make your scale read in hundred thousands, and then omit 5 zeros from each of the populations.)

4. The longest suspension bridges are: Golden Gate, 4200 ft.; George Washington, 3500 ft.; San Francisco, 2300 ft.; Ambassador (Detroit), 1800 ft.; Delaware River (Philadelphia), 1700 ft.; Bear Mountain (Hudson), 1600 ft.; Williamsburgh (N.Y.), 1600 ft.; and Brooklyn, 1600 ft.

5. The weight in pounds needed to crush 1 cu. in. of these stones is: Sebastopol limestone, 1100; Vermont marble, 13,000; Caen limestone, 3600; granite, 16,000; and Berea sandstone, 9000.

6. The number of children killed and injured in New York in one year was: crossing streets not at crossings, 2373; crossing against traffic lights, 1003; stealing rides, skating, riding on bicycles, etc., 1129; playing games in street or running into street, 2913; stepping from behind parked cars, 460; collision of vehicles, 1362; all other causes, 1783.



©National Bureau of Casualty and Surety Underwriters.

A DANGEROUS SPORT

Coasting on city streets costs the lives of thousands of children every year. Play in the parks and play safe.

In what way was the largest number of children injured or killed? The second largest number?

What was the total number of children killed in New York that year? How can boys and girls help to avoid such a loss of life?

7. The largest navies, estimated in the number of tons, are: U.S., 1,000,000; Great Britain, 1,200,000; Japan, 800,000; France, 600,000; and Italy, 400,000.

8. The principal crops of the United States in millions of acres are: wheat, 47; corn, 102; oats, 36; barley, 10; hay, 66; and cotton, 30.

9. The census shows that 8% of families consist of only 1 person, 23% of 2 persons, 21% of 3 persons, 17% of 4 persons, 12% of 5 persons, 7% of 6 persons, and 12% of more than 6 persons.

10. The lengths of the longest rivers are: Mississippi, 4200 mi.; Amazon, 3900 mi.; Amur, 2900 mi.; Lena, 2800 mi.; Nile, 4000 mi.; Congo, 2900 mi.; Yangtze, 3100 mi.; and Ob, 3200 mi.

11. The cost of educating each pupil per year in certain cities is: Boston, $130; Chicago, $95; Cleveland, $97; Minneapolis, $89; New Orleans, $53; New York, $150; Salt Lake City, $70; San Francisco, $129; Washington, $118; and Yonkers, $158.

If you go to school about 160 days a year, how much money is being spent per day on your education? Are you doing your part so that you get that much value from it?

12. Make a bar graph for the number of crimes committed in New York City, and shade the part of the bar that represents the number of arrests.

Kind of Crime	*Cases*	*Arrests*
Murder.......	1396	711
Robbery......	1251	921
Burglary......	2980	1506
Assault.......	2457	2142
Other felonies..	2681	2323
Misdemeanors .	2702	2595

About what fraction of those committing robbery is captured? Assault? Misdemeanors? Do you think that crime is a safe way to get rich?

13. Make a bar graph for these: A large store offered these shoes at a sale.

	3½	*4*	*4½*	*5*	*5½*	*6*	*6½*	*7*	*7½*	*8*	*8½*	*9*
A.....		2	7	21	48	50	58	50	51	48	26	9
B.....	11	33	41	60	81	94	88	81	54	46	34	10
C.....	2	11	22	30	36	39	34	35	18	10	1	

14. Here are the world's largest cities. Make a bar to represent the metropolitan area, and shade the part that represents the city proper.

	City Proper	*Metropolitan Area*
New York.....	6,900,000	10,900,000
London.......	4,400,000	8,200,000
Tokyo........	2,000,000	5,300,000
Chicago.......	3,400,000	4,400,000
Berlin........	3,900,000	4,300,000
Paris.........	2,900,000	3,800,000
Moscow.......	2,600,000	3,700,000
Philadelphia...	1,900,000	2,900,000

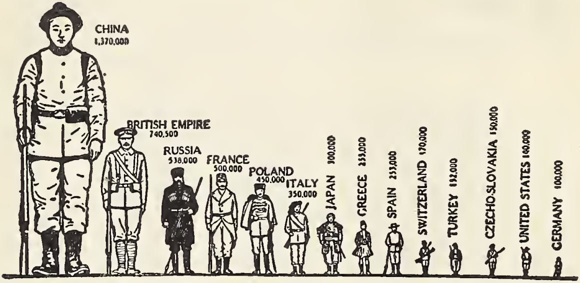
15. Make a double bar graph to compare the death-rates of the principal countries for 1920 and 1930.

	1920	*1930*
United States..	13.0	11.3
England.......	12.4	11.4
France........	17.2	15.7
Germany......	15.1	11.1
Italy..........	18.8	13.8
Japan........	25.4	18.2
Spain.........	23.8	17.4

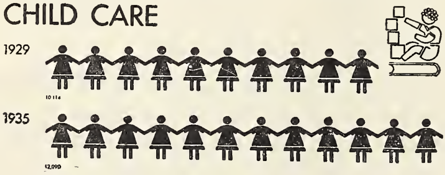
Which countries have low death-rates?

The pictogram. Shortly after the World War this picture graph was shown by *The Literary Digest.* It represents the sizes of the armies of the principal nations of the world as they were at that time. Such a graph is called a ***pictogram.*** Its advantage is that it appeals to people who would not look at a bar graph.

The size of the army is represented only by the height of the figure. For that reason it is very deceptive, for we ordinarily think of a person as three-dimensional, and a figure twice as high as another gives the impression of being 8 times as large. You know a six-foot man is far stronger than two three-foot boys. The big fellow representing China appears strong enough to sweep away all of the little fellows. In reality, however, he represents a force only about equal to that of the British Empire and Russia combined. This type of graph is used extensively in advertising for one or both of two reasons: (1) because it has an appeal for the mass of the people, and (2) because the unscrupulous advertiser can give the impression of a greater superiority over his competitors than he actually possesses.



A better type of pictogram is that shown here; all the figures are the same size, but the comparison is shown by the number of figures. Here two figures appear twice as large as one, and consequently the graph gives a more accurate picture of the facts.



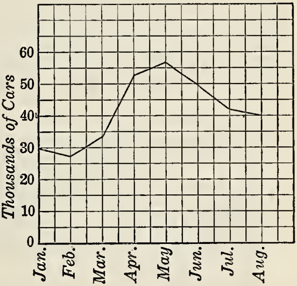
Each child represents 1000 children

How many children does this graph indicate for 1929? For 1935? How much greater is the number in 1935 than that in 1929?

THE BROKEN-LINE GRAPH

If, instead of drawing in the bars in a bar graph, you had simply marked a point where the top of each bar would reach and had then joined these points in order with straight lines, you would have had a broken-line graph.

Let us make a broken-line graph from these statistics. The Excelsior Motor Car Company sold 30,000 cars in Jan., 28,000 in Feb., 34,000 in Mar., 52,000 in Apr., 56,000 in May, 50,000 in Jun., 42,000 in Jul., and 40,000 in Aug.



First draw a horizontal line and a vertical line meeting at the lower left-hand corner of the paper. These lines are called the horizontal and vertical ***axes,*** and the point where they meet is the ***origin.*** Next we must choose suitable scales.

1. As there are 16 squares along the horizontal axis and we have 8 months to place, how many squares apart can we place them?

2. We have 15 squares along the vertical axis and the largest number is 56,000, so we should let each square represent what number? We can then omit the 3 0's by writing at the left "thousands of cars" so that 22 will mean 22,000.

3. On what vertical line shall we put the point that represents the January sales?

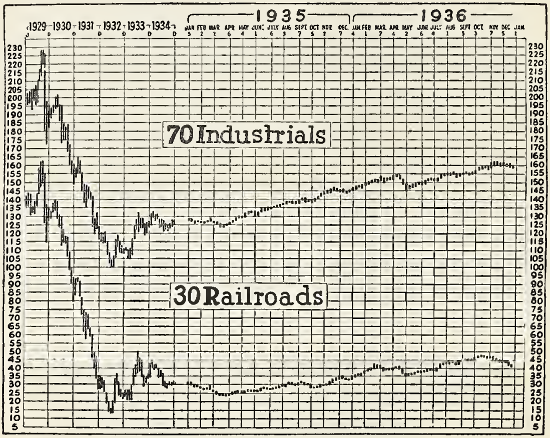
4. If each square represents 5000, how many squares up must we count to represent 30,000?

5. Where shall we put a point to represent 28,000 in February?

6. How far up shall we put the point to represent the 42,000 in July? Will this be a whole number of squares?

STOCK-MARKET GRAPHS

In the stock market, because of the fact that prices go up and down very rapidly, often many times a day or week, a type of graph is used that shows the range through which the stock varied for that day or week. The lower end of the bar indicates the lowest price of the stock and the upper end the highest price. Sometimes a cross-bar is added to show the final price of the day.



Price Trend, New York Stock Exchange.

1. When were the stocks the highest?
2. Can you tell from this graph when the depression began?
3. Which were higher, railroad or industrial stocks?
4. What was the lowest price the railroad stocks reached? When?
5. What was the price of industrials on June 1, 1935?
6. How much did industrial stocks drop from the highest point in 1929 to the lowest point in 1932?
7. What were the highest and lowest prices of industrial stocks in 1929?

Exercises

Make a broken-line graph to illustrate the data in each of these exercises.

1. Charles received these test marks in algebra: 75, 60, 80, 70, 75, 80, 45, 70, and 90.

2. Dorothy saved her money in the school bank. At the end of each month her balance was: Oct., $2.10; Nov., $3.80; Dec., $.90; Jan., $1.70; Feb., $2.80; Mar., $2.20; Apr., $1.50; and May, $2.60.

3. In the U.S., the number of people killed by automobiles was: 1923, 14,000; 1925, 18,000; 1927, 21,000; 1929, 27,000; 1931, 30,000; 1933, 29,000; and 1935, 34,000.

4. In a certain year the number of inches of rainfall in Memphis was: Jan., 4.8; Feb., 4.4; Mar., 5.3; Apr., 4.8; May, 4.2; Jun., 3.6; Jul., 3.0; Aug., 3.4; Sept., 2.8; Oct., 2.7; Nov., 4.2; and Dec., 4.5.

5. If all the money in the U.S. had been divided equally in the years given, each person would have had: 1914, $1900; 1916, $2500; 1918, $3800; 1920, $4600; 1922, $2900; 1924, $3000; 1926, $3100; 1928, $3000; 1930, $2600; 1932, $1900; and 1934, $1400.

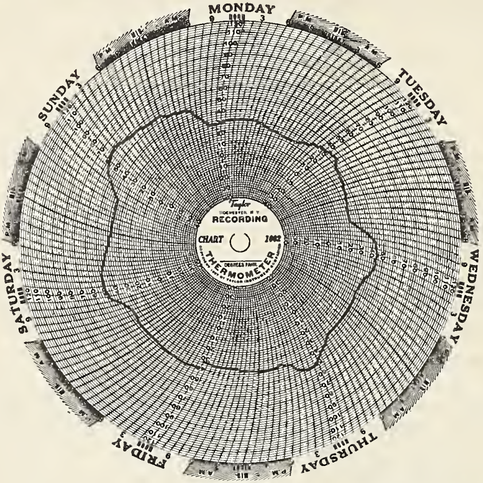
6. Mr. King's electric-light bill shows that he used the following number of kilowatt hours of electricity: Jan., 26; Feb., 24; Mar., 20; Apr., 16; May, 14; Jun., 11; Jul., 12; Aug., 15; Sept., 19; Oct., 20; Nov., 23; and Dec., 27.

7. According to the New York Police Department the number of juvenile delinquents (children committing crimes) arrested was: 1930, 7114; 1931, 6332; 1932, 6264; 1933, 6269; and 1934, 4849.

8. Make two broken-line graphs on the same axes to compare the average monthly temperatures at Chicago and Atlanta for a certain year.

	Jan.	*Feb.*	*Mar.*	*Apr.*	*May*	*Jun.*	*Jul.*	*Aug.*	*Sept.*	*Oct.*	*Nov.*	*Dec.*
Chicago..	24	26	35	47	58	67	72	72	65	54	40	29
Atlanta..	44	45	52	61	70	76	78	77	72	63	52	45

In what month was the temperature the highest? The lowest? In what month was the difference in temperatures at Atlanta and Chicago the greatest? The least? Is the difference the greatest in the summer or in the winter?



Courtesy Taylor Instrument Co.

THE RECORDING THERMOMETER

This graph shows the temperatures for a whole week. The graph paper turned slowly by clockwork and a pen moved up and down on it according as the temperature changed.

9. The exports and imports of the United States in millions of dollars were:

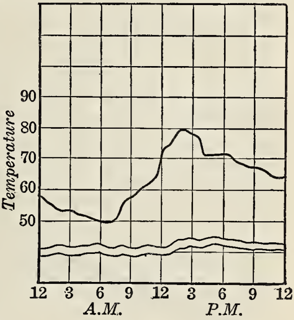
Year.....	1860	1870	1880	1890	1900	1910	1920	1930
Exports...	400	530	835	858	1395	1745	8228	3843
Imports...	362	462	668	789	850	1557	5279	3061

10. One week the attendance in an algebra class was: Mon. 28, Tues. 32, Wed. 29, Thurs. 30, and Fri. 27.

HISTORICAL NOTE. The term *statistics* comes from *state.* It was so called because early statistics dealt with state affairs and were collected by the government.

SMOOTH-CURVE GRAPHS

This graph is called a smooth-curve graph. It records the temperatures for a certain day. Compare this graph with the broken-line graph on page 9. Why do you suppose we use a smooth curve here and a broken line there? Recording thermometers make graphs like this. When you look at an ordinary thermometer, you can tell what the temperature is, but you cannot tell what it was at 6:30 this morning. The advantage of the graph is that you can tell what the temperature was at any time in the past.



1. Was there a time between January and February when the number of motor cars sold was half way between 30,000 and 28,000, or does the number jump abruptly from one to the other? Was there a time between 6 A.M. and 9 A.M. when the temperature was half way between 50° and 58°, or did it jump suddenly?

2. Do you think the smooth curve more appropriate when the quantity changes suddenly or when it changes gradually?

3. What was the temperature at 6 A.M.? At 10 A.M.? At 2 P.M.?

4. When was the temperature highest? When lowest?

5. About what time was it 70°?

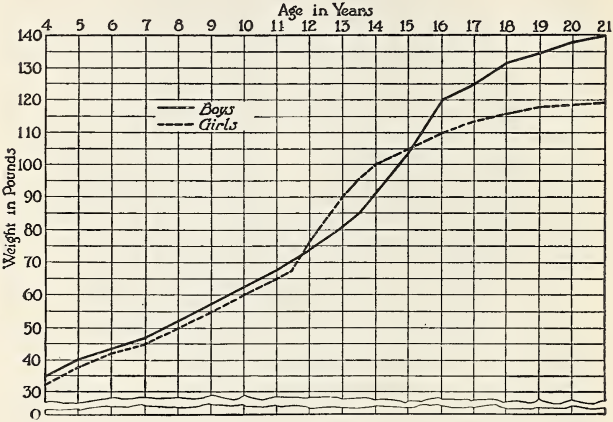
6. At what time was the temperature rising most rapidly? Falling most rapidly? During what interval was it most nearly stationary?

7. Judging by this graph alone, would you predict that the following day would be warmer or colder? Why?

8. How many degrees did the temperature rise between 9 A.M. and 12 noon? Between noon and 2 P.M.?

9. Why do you suppose a part of the graph below 50° is made to appear torn out?

This graph shows the weights of boys and girls from 4 to 21 years of age. Notice how, instead of showing the whole scale from 0 to 30, it appears as if a part had been torn out. This is a better form than simply beginning at 30.



1. How much heavier is a boy than a girl at the age of 6? At 16?

2. Are boys heavier than girls at all ages?

3. Which is gaining weight more rapidly at 12? At 14? At 16? At 20?

4. Between what ages is the girl the heavier?

5. What is the average weight of a 13-yr.-old girl?

6. If a normal boy weighs 60 lbs., about how old is he?

7. About how much difference is there between the weights of boys and girls from 6 yrs. to 10 yrs.?

8. A girl of 11 weighs 70 lbs. Is she heavier or lighter than the normal girl of her age?

9. Is a girl growing faster from 9 to 11 or from 19 to 21?

10. When does a boy grow most rapidly?

11. At what age is a normal girl twice as heavy as she was at the age of 7 yrs.?

Exercises

Illustrate these figures with a smooth-curve graph. Always lay off time on the horizontal axis.

1. The temperature was 50° at 6 A.M., 47° at 8 A.M., 53° at 10 A.M , 64° at noon, 70° at 2 P.M., 68° at 4 P.M., 62° at 6 P.M., and 54° at 8 P.M.

2. The number of persons per hundred thousand who died of tuberculosis was: 1912, 125; 1915, 123; 1918, 107; 1921, 86; 1924, 78; 1927, 71; 1930, 63; and 1933, 53.

3. Helen had a fever. The graph made by the nurse showed: Fri. A.M., 102°; P.M., 103.5°; Sat. A.M., 101.5°; P.M., 103°; Sun. A.M., 99°; P.M., 100.5°; Mon. A.M., 98.5°; and P.M., 98.5°.

4. The population of Hartford was: 1870, 37,000; 1880, 42,000; 1890, 53,000; 1900, 79,000; 1910, 99,000; 1920, 138,000; and 1930, 164,000.

5. The population of the United States in millions was: 1790, 4; 1800, 5; 1810, 7; 1820, 10; 1830, 13; 1840, 17; 1850, 23; 1860, 31; 1870, 39; 1880, 50; 1890, 63; 1900, 76; 1910, 92; 1920, 106; and 1930, 123.

From your graph estimate the population in 1925.

Make two graphs on the same axes for each of the following exercises.

6. These populations of Springfield and Seattle are in thousands.

	1870	*1880*	*1890*	*1900*	*1910*	*1920*	*1930*
Springfield	23	33	44	62	89	130	150
Seattle	1	4	43	81	237	315	366

7. Compare the interest at 6% on $100 at simple interest and at compound interest.

Years	0	5	10	15	20	25
Simple interest ...	0	30.00	60.00	90.00	120.00	150.00
Compound interest	0	33.82	79.08	139.65	220.71	329.18

(*a*) Find the simple interest from your graph for 7 yrs. For 11 yrs. By extending the graph, for 28 yrs.

(*b*) Find the compound interest for 3 yrs. For 8 yrs. For 18 yrs.

(*c*) Find how much the compound interest exceeds the simple interest for 12 yrs. For 18 yrs.

(*d*) At about what time has the compound interest become double the simple interest?

8. Here are the heights in inches of boys and girls.

Age	6	7	8	9	10	11	12	13	14	15	16
Boys	44	46	48	50	52	55	57	60	62	65	67
Girls	43	45	48	50	53	56	58	60	61	62	64

(*a*) Compare your graph with that for weights shown on page 14. Are girls taller than boys during the same years in which they are heavier than boys?

(*b*) At what age are girls growing faster than boys?

(*c*) Between what ages are girls taller than boys?

(*d*) When do boys begin to grow faster than girls do?

Depressions

In 1929 millions of poor people lost their savings by speculating in stocks just before the depression began. They did not expect a depression; yet if they had studied a graph like this, they would have known that boom times and depressions follow each other with a fair degree of regularity. One can never tell just when a depression will begin, but he can make a reasonable guess as to about when it is due.

This graph shows all depressions and prosperous times since the early days of the United States.

1. How does the length of depression compare with that of prosperity? From 1790 to 1820 about what part of the time did we have depression?

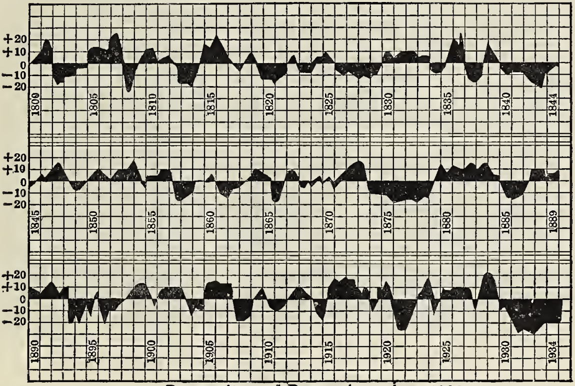
2. How many years did the depression of 1873 last? The depression of 1840?

3. Was the good time preceding the depression of 1907 longer or shorter than the depression?

4. Which depression was the worst? Next to the worst?

5. Was the country prosperous or not in 1925? 1894? 1874? 1905?

6. What condition from 1914 to 1919 would have led you to expect a depression about 1920? If you had this graph and your present knowledge in 1929, would you have expected a severe depression? Why?



Prosperity and Depressions since 1800

7. Do depressions occur with exact regularity? Every few years?

8. Do you think there will ever be another depression? Give a reason for your answer.

STATISTICS

Have you ever noticed a very good bargain in hats, or collars, or shoes in a store window, only to learn on entering that your size was not to be obtained? The merchant had to sell these odd sizes cheap and lose money on them because he did not know when he bought them that few people wore those sizes and that there was so little demand for them. If he had known something about statistics, he might have avoided that loss.

It is a peculiar fact that most things in nature, such as heights or weights of people of a certain age, sizes of collars, hats, or shoes, marks in algebra or English, diameters of trees, the time it takes runners to go 100 yds., the number of quarts of milk cows will give, and many other things, follow a very definite law. If you know this law, you can predict in advance how many of a group will be of a certain size or speed or get a passing mark. The study of these facts and the collection and grouping of them are called *statistics*.

Courtesy of International Business Machines Corp.

A STATISTICAL DEPARTMENT USING ELECTRICAL ACCOUNTING MACHINES

Every big corporation has a large statistical department that collects and organizes information about business. It prepares tables and graphs for executives to study.

How to interpret statistics.

Problem: The same test was given to two classes in algebra. In Class A, 1 pupil got 100; 2, 90; 8, 80; 12, 70; 6, 60; 4, 50; and

2, 40. In Class B, 1 got 90; 5, 80; 10, 70; 11, 60; and 3, 50. Which class did the better work?

Since it is difficult to tell from looking at the numbers, we might find their average. The ***average*** of a set of numbers is found by dividing their sum by the number of numbers.

Let us arrange the data in a column.

CLASS A

No. of Pupils	*Mark*	*Total of Marks*
1	100	100
2	90	180
8	80	640
12	70	840
6	60	360
4	50	200
2	40	80
35		2400

$$\textit{Average} = \frac{2400}{35} = 68+$$

CLASS B

No. of Pupils	*Mark*	*Total of Marks*
1	90	90
5	80	400
10	70	700
11	60	660
3	50	150
30		2000

$$\textit{Average} = \frac{2000}{30} = 67-$$

So on the basis of average, it appears that Class A is the better.

But an easier, and sometimes better, method is to find the median score. The ***median*** number is the middle number when the numbers are arranged in order of size. If there is an even number of numbers, the median is half way between the two middle numbers. One advantage that the median has over the average is that it is not so much affected by a number of exceptional size.

In Class A there are 35 pupils, so if we divide by 2 getting $17\frac{1}{2}$, we see that there are 17 pupils on each side of the middle. Consequently the 18th is the middle one. Beginning at the top, 1 and 2 are 3, and 8 are 11, and we need 7 more to make 18. This will not take all of the next 12, so the 18th pupil is among the 12. The median mark for

Class A then is 70,* the mark that each of these 12 pupils got.

In Class B there are 30 pupils, so dividing by 2, we find that there are 15 on each side of the middle. The median then is between the 15th and the 16th. This would place the median pupil in the group of 10, and the median mark is again 70.

So on the basis of the median, the classes are equal in ability.

There is another method of comparing classes, by means of the mode, which is easier to find than the median. The *mode* is the number that occurs the greatest number of times. It is the most stylish or most fashionable number.

In Class A the mark 70 occurred 12 times. Since no other mark occurred so many times, 70 is the mode. In Class B, 60, that occurred 11 times, is the mode. On the basis of the mode, therefore, Class A is the better class.

Exercises

In these exercises find the average, the median, and the mode.

1. In his 5 major subjects a student received the marks 70, 70, 75, 90, and 95.

2. The girls of Excelsior Camp hiked on successive days 8 mi., 6 mi., 5 mi., 3 mi., 6 mi., and 4 mi. First rearrange the data in order of size.

3. On a test 3 pupils got 60%; 11, 70%; 8, 80%; 6, 90%; and 2, 100%.

4. A class was asked to contribute to charity; 15 pupils gave 10¢ each; 8, 15¢ each; 3, 25¢ each; and the teacher gave $3. Do you think the average or the mode better represents the facts?

5. In the final examination 5 pupils received 100%; 20, 90%; 27, 80%; 22, 70%; 20, 60%; 8, 50%; and 3, 40%.

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* For simplicity we are assuming that these 12 marks are all 70's and not distributed through the interval 70 to 80.

6. Charles sells newspapers. On Mon. he sold 46; on Tues., 40; on Wed., 53; on Thurs., 51; on Fri., 40; and on Sat., 64.

7. In weekly tests one term Gladys got marks of 8, 10, 7, 10, 9, 4, 8, 9, 8, 5, 9, 7, 6, 8, 6, 3, 10, and 8.

8. On a single day a telephone exchange handled 80 calls at 5¢ each, 20 calls at 10¢ each, 6 calls at 20¢ each, and 1 call at $2.40.

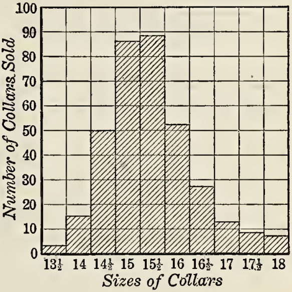
9. In a class test 2 pupils got A, 8 got B, 13 got C, 7 got D, and 2 got F. Find the median and mode.

10. In settling how much damages a defendant should pay, a jury decided to have each member write on paper the amount he thought satisfactory. Here are the results: $75, $200, $200, $2400, $200, $50, $250, $300, $200, $250, $200, and $275. Which would represent better the opinion of the jury, the average or the median? The average or the mode?

THE FREQUENCY POLYGON OR HISTOGRAM

If we arrange the individuals of a group in order of weight, height, etc. and make a bar graph showing the number of them in each division, the figure is called a *frequency polygon,* or *histogram.* In making this polygon the bars are placed vertically and are put together without leaving spaces between them.

The number of collars of the different sizes sold one year by a merchant is shown in this frequency polygon.



1. What size of collar had the greatest sale? The second largest sale? The third largest sale?

2. If you were buying collars for a store, would you buy more of the middle sizes or of the very large or very small sizes?

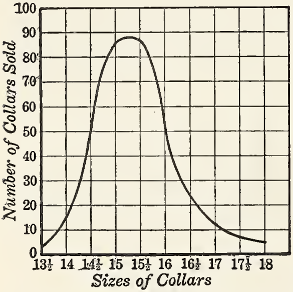
3. Which can you determine more easily from a frequency polygon, the average, the median, or the mode?

4. If a merchant bought about the same number of collars of each size, of what sizes would you expect to find the largest numbers left on hand?

5. Do many men wear collars of size 13½? Of size 18? Of size 15?

6. How does the number of men wearing size 14½ compare with the number wearing size 16?

If, instead of drawing the bar graph as in the figure above, we had plotted the points and had drawn a smooth-curve graph through them, we would have the figure shown at the right. This graph is called a ***normal frequency curve.*** If the number of objects is very large, the curve will have the same height on both sides of the median line.



Is the highest point of the curve near the middle or near an end? Can you draw a line along which the figure might be folded so that the part on one side would almost fit that on the other side?

Exercises

Make a frequency polygon for each of these exercises:

1. In the U.S. army intelligence is rated A, B, C+, C, etc. The per cent of white men in each group is: A, 4%; B, 8%; C+, 15%; C, 25%; C−, 24%; D, 17%; and D−, 7%.

2. The intelligence quotients for seniors in a high school were:

IQ..........	76	81	86	91	96	101	106	111	116	121	126
No. of pupils..	23	36	77	138	192	211	179	131	82	41	22

3. In a test in handwriting given in the eighth grade, 3% of the pupils got 30; 10%, 40; 16%, 50; 22%, 60; 23%, 70; 23%, 80; and 3%, 90.

4. In a certain city the number of high-school seniors, distributed by age, was:

Age	15	16	17	18	19	20	21	22
No. of pupils	80	1420	3120	4100	1950	980	120	80

5. The heights of freshmen in a certain college were:

Height in inches	60–61	62–63	64–65	66–67	68–69	70–71	72–73	74–75
No. of men	18	70	245	590	890	640	230	68

6. A large department store furnished these figures. The number of shirts of a particular design sold one spring, grouped by sleeve length on the left and neck-band size on top, was:

Sleeve Length	*Neck-Band Sizes*									
	13½	*14*	*14½*	*15*	*15½*	*16*	*16½*	*17*	*17½*	*18*
32	1	13	26	23	20	12	0	0	0	0
33	6	16	83	98	67	32	6	5	0	0
34	3	12	30	54	61	26	15	4	5	3
35	0	4	20	32	26	26	12	8	4	3
36	0	1	2	5	8	4	5	2	0	1
Totals	10	46	161	212	182	100	38	19	9	7

Notice the central tendency, that is, that the numbers grow larger as you approach the middle, for all horizontal rows of figures and also for the vertical columns.

(*a*) Make a frequency polygon using the neck-band size on the horizontal axis and the total number sold on the vertical axis.

(*b*) In the same way make frequency polygons for the number sold at each of the different sleeve lengths.

(*c*) Make a frequency polygon using the sleeve length on the horizontal axis and the number of $15\frac{1}{2}$-size shirts sold on the vertical axis.

Thought Questions

1. Why is it important that the scale on the bar graph should begin at 0, whereas on the smooth-curve graph it may begin at any convenient number? What effect would beginning at some other number have on the length of the bars? On the comparative length of the bars? On the shape of the smooth-curve graph?

2. Does a set of numbers always have a median? A mode? Which of the three, average, median, or mode, do you think the most satisfactory for comparing your final marks with those of other pupils? Why?

3. Which do you think would give the more regular normal frequency curve, and why: (*a*) Your final marks in all of the subjects you take this term? (*b*) The final marks in algebra of all the pupils in a large city school?

Review Exercises

Decide what type of graph is most appropriate for each of these exercises and draw the graph:

1. The number of million telephones in the United States was: 1902, 2; 1907, 6; 1912, 9; 1917, 12; 1922, 14; 1927, 19; 1932, 17.

2. The amount of food, that cost $1 in 1926, cost as follows:

1913	*1916*	*1919*	*1922*	*1925*	*1928*	*1931*	*1934*
$.64	$.76	$1.29	$.88	$1.00	$1.01	$.75	$.60

3. The population of the earth in millions by continents is: Asia, 950; Europe, 550; America, 230; and Africa, 150.

4. The number of millions of acres used for crops was: wheat, 48; corn, 102; oats, 36; barley, 10; hay, 66.

5. One year the exports of certain countries in hundred millions of dollars were: United States, 49; France, 21; Great Britain, 40; and Germany, 24.

6. The assets and liabilities of a corporation were:

	1910	*1915*	*1920*	*1925*	*1930*	*1935*
Assets	85,000	115,000	102,000	140,000	165,000	110,000
Liabilities . .	62,000	80,000	78,000	90,000	103,000	85,000

When was the firm most prosperous? When least prosperous?

7. One day the temperature readings in a certain town were:

Hour	8	9	10	11	12	1	2	3	4	5	6
Temperature	10°	12°	16°	23°	32°	40°	43°	43°	41°	34°	29°

(*a*) From your graph determine the temperature at 11.30 A.M.; at 4.30 P.M.

(*b*) Between what hours was the temperature rising most rapidly? Falling most rapidly?

8. Here is a table of the weekly wages of boys who leave school at 14 and boys who leave school at 18.

Age	*Wages of Boy Who Left School at 14*	*Wages of Boy Who Left School at 18*
14	4	
16	5	
18	7	10
20	9	15
22	11	20
24	12	24
26	13	30

How does the pay of the boy who left school at 14 compare with that of the boy who remained in school until 18 when the two boys are 24? 30? Do you think that education pays?

9. The number of people per square mile in the United States was:

1870	*1880*	*1890*	*1900*	*1910*	*1920*	*1930*
13	17	21	26	31	35	41

10. Lucy had a fever. The nurse recorded these temperatures. (The normal temperature is 98.6°.) Tues. A.M., 100°; P.M., 102°; Wed. A.M., 100.5°; P.M., 103.5°; Thurs. A.M., 102°; P.M., 104°; Fri. A.M., 100.5°; P.M., 102.5°; Sat. A.M., 99°; P.M., 99.5°.

11. Arrange these classes in order of rank, make a frequency polygon of each, and find its average, median, and mode. Then from your results determine which class is better.

Class 1				Class 2			
Name	*Mark*	*Name*	*Mark*	*Name*	*Mark*	*Name*	*Mark*
JBA	75	ARN	80	FLB	70	JBP	60
CHB	65	RNP	60	HHC	80	FJP	90
RMB	80	DMR	100	AME	95	JHR	50
ELC	70	KHR	85	JOF	70	FSR	75
JND	85	RPR	85	FDG	85	WJS	80
RDD	95	AKS	80	WBG	60	HNS	70
WHF	75	RWS	70	MJH	75	RAS	65
LBH	60	CLS	50	RWJ	65	WBT	80
RFH	90	JSS	80	CBL	90	CWV	95
JWL	80	HCT	85	FSL	80	KJW	55
JJM	55	JSW	80	SHM	85	HPW	60
WSM	95	MRW	85	LSM	60	RRW	80
FWM	80	KSY	75	RMN	70		

12. A grocer's record of sales of peaches for a certain year showed that at $1.25 a basket he sold 10 baskets; at $1.00 a basket he sold 30 baskets; at $.75, 70 baskets; at $.60, 85 baskets; and at $.50, 90 baskets. Draw the graph, and from it estimate the probable demand when the price is $1.10 a basket.

13. The temperatures recorded for a certain day were:

Hour	8	9	10	11	12	1	2	3	4	5	6
Temperature	50	52	56	63	72	80	83	83	81	74	69

14. This table shows the earnings and spendings of a boy from Apr. to Oct.

	Apr.	*May*	*Jun.*	*Jul.*	*Aug.*	*Sept.*	*Oct.*
Earnings	$2.00	$3.20	$4.80	$10.40	$12.00	$4.40	$5.20
Spendings	3.20	3.60	2.40	2.00	1.60	8.40	4.00

(*a*) Read the points where the graphs cross and tell what they mean.

(*b*) Did the boy spend his vacation at work or at play? How is this shown by the graph?

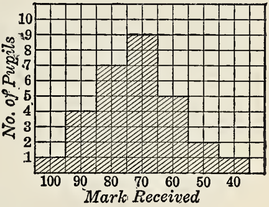
Matching Test

Write the numbers from 1 *to* 6 *in a column. After each number, write the letter before the word that corresponds to the sentence after the number.*

1. The name of the number obtained by dividing the total by the number of items.	A. Median.
2. The kind of graph to use when the data change gradually from one value to the next.	B. Mode.
3. The name of the number that occurs the largest number of times.	C. Average.
4. The kind of graph to use when we are comparing the sizes of different things at the same time.	D. Smooth-curve.
5. The name of the number that is equally distant from both ends when the data are arranged in order of size.	E. Bar graph.
6. The kind of graph to draw when the data change abruptly.	F. Broken-line graph.

Interpreting Graphs Test

1. This frequency polygon shows the marks the pupils of a class received in a test in algebra. *Copy the sentences, inserting the correct word in the blank spaces.*



(*a*) The number of pupils getting 80% was . . . .

(*b*) The mode was . . . .

(*c*) Seven pupils received a mark of . . . .

(*d*) If 60% was the passing mark, the number of pupils who failed was . . . .

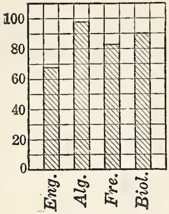
2. This graph (page 28) shows a girl's final marks in English, algebra, French, and biology.

(*a*) She received the lowest mark in . . . .

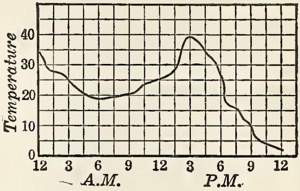
(*b*) She received the highest mark in . . . .

(*c*) Her mark was about 90% in . . . .

(*d*) Her mark in algebra was . . . .



Ex. 2



Ex. 3

3. This graph shows the temperatures at Denver for a single day, as shown by a recording thermometer.

(*a*) At 6 A.M. the temperature was about . . . .

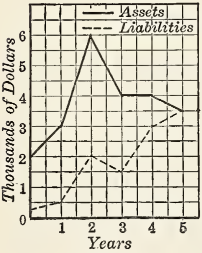
(*b*) The temperature rose above 32° at about . . . o'clock.

(*c*) The temperature was rising most rapidly about . . . .

(*d*) Was this graph recorded in January or July? . . .

(*e*) From the information given by this graph alone, would you predict that the following day would be warmer or colder? . . .

(*f*) The temperature was highest at about . . . o'clock.



(*g*) The temperature at noon was about . . . .

(*h*) The temperature was falling most rapidly about . . . .

4. This graph shows the assets and liabilities of a business firm for a period of 5 years.

(*a*) The firm was most prosperous the . . . year.

(*b*) The assets at the end of the fourth year were . . . .

(*c*) If capital equals assets minus liabilities, at the end of the first year the capital was . . . .

(*d*) During the fifth year the capital reduced to . . . .

Chapter 2

THE FORMULA

WE all like short methods of expressing ourselves. We cut down "cannot" to "can't," "automobile" to "auto." We use abbreviations, as for example, S.D. for South Dakota, C.O.D. for cash on delivery, U.S.A. for United States of America, T.R.H.S. for Theodore Roosevelt High School, and so on. And we study shorthand so as to be able to write more words per minute. Now in algebra we shall learn many short cuts for processes that were long and clumsy in arithmetic, and the most important of these short cuts is the formula. We shall now discover what a formula is.

Investigating the formula. How would you find the area of a rectangle if its length were 12 and its width were 10?

What procedure did you follow in getting your answer?

But this rule is long. Could you abbreviate it by writing only the initial letters of the important words, omitting the other words altogether, and using such signs as =, ×, etc.? If you have succeeded in this, you have a formula, for:

A formula is a shorthand way of writing a rule.

How to make a formula.

Rule. The area of a rectangle equals the product of its length and width.

Using only the important words, we have:

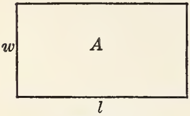
$$\text{Area} = \text{length} \times \text{width}$$

Now we can shorten it still more by writing only the initial letters of these words: $A = l \times w$

And if we agree that, whenever two letters are written together with no sign between them, we are to understand that the product of the quantities is meant, we can still further shorten our rule by omitting the times sign. Our rule now becomes

$$A = lw$$

This we call a *formula.*



Rectangle

In this formula, l, w, and A all represent numbers. If the rectangle is 8 in. long and 6 in. wide, l represents the number 8, w represents 6, and A represents 48, for:

$$\begin{aligned} A &= lw \\ &= 8 \times 6 \\ &= 48 \end{aligned}$$

Exercises

1. Using the formula, $A = lw$, find the value of A if:
- (*a*) l is 7 and w is 3
- (*b*) l is 14 and w is 3
- (*c*) l is 21 and w is 3
- (*d*) l is 7 and w is 6

2. How does the value of A change when:
- (*a*) l is doubled but w remains the same?
- (*b*) l is trebled but w remains the same?
- (*c*) l remains the same but w is doubled?

3. Using the same formula, find A when:
- (*a*) $l = 384$ and $w = 226$
- (*b*) $l = 17\frac{1}{2}$ and $w = 11\frac{1}{4}$
- (*c*) $l = .0013$ and $w = .0007$

4. Can l stand for any number you wish it to? Can w? Can A? Can all three of these letters at the same time stand for any three numbers you can select for them?

Make formulas for these rules:

5. The diameter of a circle is twice the radius.

6. The distance a car travels is the product of the rate and the time.

7. The number of inches (i) is 12 times the number of feet (f).

8. The cost of a piece of meat is found by multiplying the price per pound by the number of pounds.

9. The number of rods (r) in a certain distance is 320 times the number of miles (m) in it.

10. The wages a man receives (w) equal the product of the price per hour (p) and the number of hours (h).

11. The volume of a box is the product of its length, width, and height.

12. The cost of sending a package by air mail is 5 times the number of ounces (z).

13. The number of seats in the room (s) is the product of the number of rows (r) and the number of seats in a row (n).

14. The cost (c) of 3¢ stamps is 3 times the number of stamps bought (s).

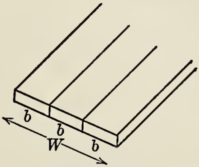
Thought Question

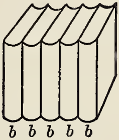
If y stands for a number of years and m for the number of months in the same period of time, which is correct:

(*a*) $y = 12\,m$? (*b*) $m = 12\,y$?

Practice in the shorthand of algebra.

Illustration. The width of each board is b. What is the width W of three such boards? We can add $b + b + b$, or we can multiply b by 3. In either case the answer is $W = 3\,b$.

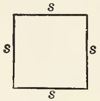


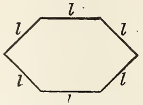


Exercises

1. If the thickness of a book is b, what is the thickness T of 5 such books standing together on a shelf?

2. Each edge of a square is s. What is the distance around the square?





3. A field has 6 equal sides each of length l. Find the length of the fence around it.

4. A stairway has 15 steps each of height h. What is the total height (H) of the stairway?

Express each of these in shorter form:

5. $p = e + e + e + e$

6. $m = 5p + 3p + 2p$

7. $M = 3w + w + 4w + w$

8. $A = ab + ab + ab$

9. $k = 5s + 3s + 4s$

10. $V = 3Bh + 5Bh + Bh$

The *perimeter* of a figure is the distance around it, or the sum of its sides.

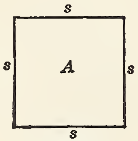
11. (*a*) How many sides has a square? If 1 side is s, what is the sum? Make a formula for the perimeter of a square whose side is s.

(*b*) Find p when $s = 5$; when $s = 10$.

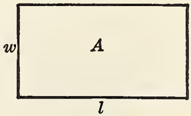
(*c*) When s becomes twice as large as before, does p become twice as large as it was?

(*d*) Find the value of p when $s = 5a$.

12. (*a*) Make a formula for the perimeter of the rectangle whose length is l and whose width is w. How many sides of length l has it? How many of width w?



Square
Ex. 11



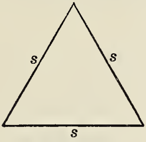
Rectangle
Ex. 12

(*b*) Find p when $l = 14$ and $w = 10$. When $l = 28$ and $w = 10$.

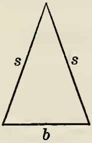
(*c*) When l becomes twice as large, does p become twice as large?

(*d*) Find the value of p when $l = 8x$ and $w = 3x$.

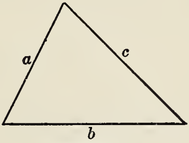
13. (*a*) Make a formula for the perimeter of an equilateral triangle whose sides are each s.



Equilateral Triangle



Isosceles Triangle



General Triangle

(*b*) Find p when $s = 4$; when $s = 12$.

(*c*) By what is p multiplied when s is multiplied by 3?

14. (*a*) Make a formula for the perimeter of an isosceles triangle whose base is b and whose other sides are each s.

(*b*) Find the value of p when $b = 10$ and $s = 18$.

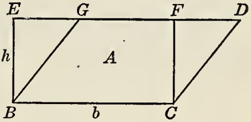
15. (*a*) Make a formula for the perimeter of a triangle whose sides are a, b, and c.

(*b*) Find p when $a = 10$, $b = 8$, and $c = 9$.

(*c*) Find p when $a = 20$, $b = 16$, and $c = 18$.

THE FORMULA IN GEOMETRY

The parallelogram. *GBCD* is a parallelogram. The sides *GD* and *BC* run in the same direction, and the sides *GB* and *DC* run in the same direction. The parallelogram has both pairs of sides parallel. We can change it into a rectangle by cutting off triangle *DCF* and replacing it by triangle *GBE*.



1. What is the area of the rectangle *EBCF*?

2. Does triangle *GBE* have the same area as triangle *DCF*?

3. Does parallelogram $GBCD$ have the same area as rectangle $EBCF$?

4. What then is the area of $GBCD$, using b for base and h for height?

The area of a parallelogram equals the product of its height and its base.

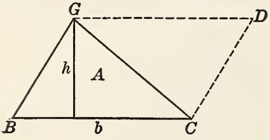
$$A = hb$$

Exercises

Find the value of A if:

1. $b = 8$ and $h = 5$
2. $b = 8$ and $h = 10$
3. $b = 8$ and $h = 15$
4. $b = 12$ and $h = 4$
5. $b = 24$ and $h = 4$
6. $b = 6$ and $h = 8$
7. What change takes place in A when h is doubled? Trebled?
8. What change takes place in A when b is doubled? Halved?
9. What is the effect on A if b is halved and h is doubled?

The triangle. 1. What is the area of the parallelogram whose height is h and whose base is b?



2. How does the area of the triangle GBC compare with that of the parallelogram?

3. Make a formula for the area of the triangle, using A for area, h for height, and b for base.

The area of a triangle equals half the product of its height and base.

$$A = \frac{1}{2} hb$$

Exercises

Using this formula, find the value of A when:

1. $h = 10$ and $b = 4$
2. $h = 10$ and $b = 8$
3. $h = 10$ and $b = 12$
4. $h = 10$ and $b = 2$
5. $h = 5$ and $b = 4$
6. $h = 20$ and $b = 4$
7. $h = 20$ and $b = 8$
8. $h = 5$ and $b = 8$

9. How is the value of A changed if h remains the same but:
(*a*) b is doubled? (*b*) b is trebled? (*c*) b is halved?

10. How is the value of A changed if:
(*a*) b remains the same but h is doubled?
(*b*) both h and b are doubled?
(*c*) b is doubled but h is halved?

The Formula in the Home

1. How many square yards of linoleum are needed for the floor of a kitchen 12 ft. long and 9 ft. wide?

2. Find the cost of paving a sidewalk 30 ft. long and 6 ft. wide at $.20 a sq. ft.

3. How many gallons of paint will be needed for the 4 walls of a building 40 ft. long, 25 ft. wide, and 20 ft. high, if 1 gal. of paint will cover 400 sq. ft.?

4. A house whose width is 30 ft. has 2 gabled ends, and the ridge is 20 ft. above the plate at the eaves. How many sq. ft. of boards are needed to cover both gables?

5. Mr. Brown buys a field 64 rds. by 50 rds. at $25 an acre. If an acre equals 160 sq. rds., what price should he pay?

Practice with Formulas

Using initial letters, make formulas for these rules:

1. (*a*) The volume V of a rectangular box equals the product of its length, width, and height.

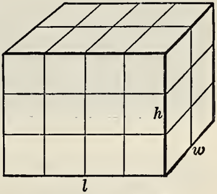
(*b*) Using your formula, find the volume of a box whose length is 12, width 8, and height 5.

(*c*) Find V when $l = 7$, $w = 5$, and $h = 3$.

2. (*a*) The distance a car goes equals the speed multiplied by the time.

(*b*) Using your formula, find the distance when the speed is 28 mi. an hr. and the time is 4 hrs.

(*c*) Find d when $s = 15$ and $t = 12$.



Rectangular Box

Photograph from Philip D. Gendreau, N.Y.

THE PYRAMIDS OF EGYPT

More than 3,000 years ago, the Egyptians used mathematics in designing and constructing these great monuments.

3. (*a*) The volume of a pyramid equals $\frac{1}{3}$ the product of its height and base.

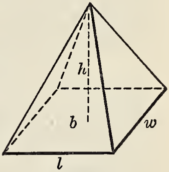
(*b*) Find V when $b = 42$ sq. in. and $h = 8$ in.

(*c*) Find V when $b = 16$ and $h = 10$.

4. (*a*) When the base of a pyramid is a rectangle, we can find the volume by taking $\frac{1}{3}$ the product of the length, width, and height.

(*b*) Using your formula, find the volume of a pyramid if its height is 300 ft. and the length and width of its base are 200 ft. and 170 ft.

(*c*) Find V when $h = 32$, $l = 18$, and $w = 10$.



Pyramid

5. (*a*) The interest on a sum of money is found by multiplying the principal by the rate by the time.

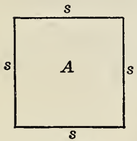
(*b*) Using your formula, find the interest on \$300 at 4% for 6 yrs.

(*c*) Find i when $p = 250$, $r = .06$, and $t = 4$.

Another short cut in writing formulas. The exponent. The square is a rectangle. Consequently its area is found by multiplying its length by its width.

$$A = ss$$

Instead of writing ss, we write s^2 (read s-square). Our formula then becomes $A = s^2$. Similarly $sss = s^3$ (read s-cube) and $ssss = s^4$ (read s-fourth).



Square

Quantities multiplied together to form a product are called *factors* of that product. A product of equal factors is called a *power* of one of the factors.

A small number written to the right and above a quantity to tell how many times it is taken as a factor is called an *exponent.*

Be careful that you do not make the mistake of thinking that the exponent 2 in s^2 means that the value of s is to be multiplied by 2. It means that the value of s is to be multiplied by itself. For example:

$$\text{If } s = 3, \text{ then } s^2 = s \times s = 3 \times 3 = 9$$
$$\text{If } s = 5, \text{ then } s^2 = s \times s = 5 \times 5 = 25$$

Can you discover a reason for calling the exponent 2 "square" and the exponent 3 "cube"?

Exercises

Write each of these expressions in the shortest form:

1. xxx
2. $yyyyy$
3. $AAAA$
4. $xxyyy$
5. $aaaaabb$
6. $xxxyzz$
7. $3 \times 3 \times 3$
8. $2 \times 2 \times 2 \times 2$
9. $4 \times 4 \times 4 \times 4 \times 4$
10. $2\,aa$
11. $3\,xxyyy$
12. $4\,xxyzzz$

If $s = 4$, find the value of:

13. s^2 **14.** s^3 **15.** s^4 **16.** $3\,s^2$

The exponent belongs only to the letter after which it is written. xy^3 means $xyyy$, not $xxxyyy$. $5\,A^2$ means $5\,AA$, not $5 \times 5\,AA$. When $A = 2$, $5\,A^2 = 5 \times 2 \times 2 = 20$.

When $r = 5$, find the value of:

17. $5\,r$ **18.** $2\,r^2$ **19.** $4\,r^3$ **20.** r^4 **21.** $2\,r^4$

Using the formula, $A = s^2$, find the value of A when:

22. $s = 6$ **23.** $s = 3$ **24.** $s = 1.2$ **25.** $s = 2\frac{1}{2}$ **26.** $s = 12$

27. How many times as great does A become when s is doubled? Halved?

Formulas for Familiar Forms

1. You learned that the volume of a rectangular box was

$$V = lwh$$

Now a ***cube*** is a rectangular box having all of its edges equal.

(*a*) Letting e represent the edge of a cube, write a formula for the volume of the cube.

(*b*) Using your formula, find the volume of a cube whose edge is 6 in.

(*c*) Find the value of V when $e = 4$; when $e = 8$.

(*d*) By what number is the volume of a cube multiplied when its edge is doubled?

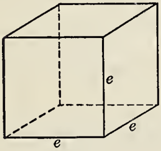
2. How many faces has a cube? Are they all equal? Are they all squares? What is the area of a square whose edge is e? Now write a formula for the total surface of a cube.

(*a*) Find the surface of a cube whose edge is 10 in.

(*b*) Using the formula $S = 6e^2$, find S when e is 4, 8, 12, 2

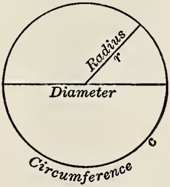
(*c*) By what number is S multiplied when e is doubled? Trebled? Halved?

(*d*) When e increases, would you say that S increases at the same rate, faster, or slower than e?



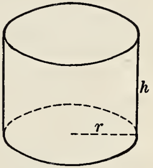
Cube

Exs. 1, 2



Circle

Exs. 3, 4



Cylinder

Ex. 5

3. The circumference of a circle is the length of the circle or the distance around the circle. The circumference of a circle is 2π times the radius, where π is $3\frac{1}{7}$ or more accurately 3.1416.

(*a*) Express this rule as a formula.

(*b*) Using $\pi = 3\frac{1}{7}$, find c when $r = 7$; $r = 14$; $r = 21$.

(*c*) By what number is the value of c multiplied when we double r? When we treble r?

4. The area of a circle equals π times the square of its radius.

(*a*) Express this area as a formula.

(*b*) Using your formula, find the area of a circle whose radius is 5 in. (π is always approximately $\frac{22}{7}$ or 3.1416.)

(*c*) Find A when $r = 4$.

5. The common vegetable or fruit can is a cylinder. Its top and bottom are circles. What is the area of its base if the radius is r? The volume of a cylindrical can is found by multiplying π by the square of the radius by the height.

(*a*) Write a formula for the volume.

(*b*) Using your formula, find the volume of a can of fruit if the radius is 2 in. and the height is 3 in.

(*c*) Using $\pi = 3.14$, find the volume of these cylinders:

$r = 2, h = 2$ $\qquad$ $r = 4, h = 2$
$r = 2, h = 4$ $\qquad$ $r = 4, h = 4$

(*d*) Which has the greater effect on the volume, doubling the radius or doubling the height?

(*e*) By what number is the volume multiplied when the radius and the height are both doubled?

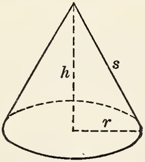
(*f*) If you were buying a can of tuna fish, would you select one whose height was twice as great or one whose radius was twice as great, other conditions being equal?

6. The volume of a cone is found by multiplying $\frac{1}{3}\pi$ times the square of the radius times the height.

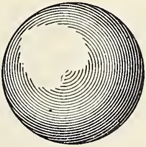
(*a*) Write this rule as a formula.

(*b*) Find the volume of a cone if $r = 7$ and $h = 10$; if $r = 14$ and $h = 10$; if $r = 7$ and $h = 20$.

(*c*) If you could have your choice in selecting an ice cream cone, would you take one whose height was twice as great or one whose radius was twice as great? Why?



Cone
Ex. 6



Sphere
Ex. 7

7. The volume of a sphere is $\frac{4}{3}\pi$ times the cube of its radius.

(*a*) Express this rule as a formula.

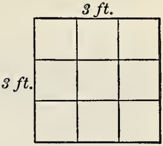
(*b*) Find the volume of a sphere whose radius is 7.

(*c*) Using $\pi = 3.14$, find the volume of a sphere if $r = 10$.

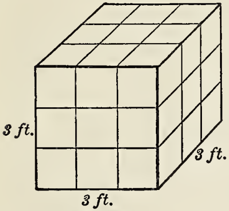
(*d*) Does the volume increase when the radius increases?

(*e*) By what number is V multiplied when r is doubled?

Square and cubic measure. Do you know the number of square inches in a square foot, or square yards in a square rod? Probably you have forgotten, but it does not matter. You do not need to remember. The formula $s = l^2$ will tell you. (s is the number of square units and l the number of units of length.)



1 sq. yd.



1 cu. yd.

Illustration. How many square feet are there in a square yard? Since there are 3 ft. in a yd., $l = 3$. Then $s = l^2$ becomes $s = 3^2 = 9$. There are 9 sq. ft. in 1 sq. yd.

In the same way by using $c = l^3$, you can make your own cubic measure table.

Exercises

Using the table at the left as a guide, complete the other tables on a separate sheet of paper.

Linear Measure

1. 12 in. = 1 ft.
3 ft. = 1 yd.
$5\frac{1}{2}$ yds. = 1 rd.

Square Measure

. . . sq. in. = 1 sq. ft.
. . . sq. ft. = 1 sq. yd.
. . . sq. yds. = 1 sq. rd.

Cubic Measure

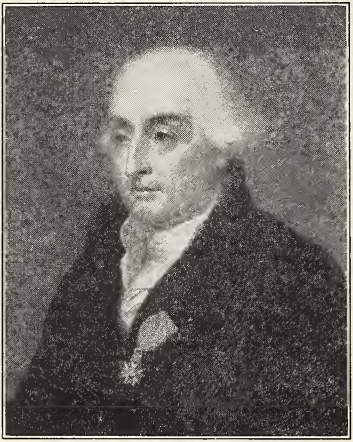
. . . cu. in. = 1 cu. ft.
. . . cu. ft. = 1 cu. yd.
. . . cu. yds. = 1 cu. rd.

2. 10 millimeters (mm.) = 1 centimeter
10 centimeters (cm.) = 1 decimeter
10 decimeters (dm.) = 1 meter (M.)

. . . sq. mm. = 1 sq. cm.
. . . sq. cm. = 1 sq. dm.
. . . sq. dm. = 1 sq. M.

. . . cu. mm. = 1 cu. cm.
. . . cu. cm. = 1 cu. dm.
. . . cu. dm. = 1 cu. M.

How to express other relations in formulas. We have learned that multiplication in algebra is generally expressed by putting two letters together with no sign between them. We may also use the ordinary times sign of arithmetic (×), or we may write a dot between the quantities half way up the height of the letter. Often the quantities to be multiplied are put in parentheses ( ) with no sign between the parentheses. Two numbers cannot be put together to indicate multiplication because they form a new number that is not the product wanted, so either the times sign, the dot, or the parenthesis must be used.



JOSEPH LOUIS LAGRANGE
(1736–1813)
French mathematician who took a prominent part in establishing the metric system of weights and measures. He also wrote on the solution of equations, but his most famous work was his *Analytic Mechanics*.

$$2 \cdot 3 \cdot 4 = 24 \qquad 2 \times 3 \times 4 = 24 \qquad (2)(3)(4) = 24$$

To express addition or subtraction, we always write the plus (+) or minus (−) sign.

$$2x + 3y \qquad 5a - b$$

To express division we can use the ordinary sign (÷) as in arithmetic, but more often we write the number we are dividing by (the divisor) under the number we are dividing it into (the dividend) with a line between them just as in fractions.

$$18 \div 6 = 3 \qquad \frac{18}{6} = 3$$

Puzzle. Here is a simple example in arithmetic. Try it.

$$2 + 2 \times 4 - 6 \div 3 \times 2 + 5 \times 2 \div 2 - 1$$

What answer do you get? The correct answer is 13. Can you discover how to get it?

It is evident that you can get a large number of different answers depending on what you do first. So if we are going to agree on an answer, we must first agree on the order in which we shall perform the operations. Mathematicians have agreed on the following order:

Order of operations. 1. First perform all multiplications.

2. Next perform all divisions from left to right.

3. Finally perform all additions and subtractions in order from left to right.

Now try the puzzle again, and see if you can get the right answer.

Need for the parenthesis. If we do not want the multiplication done first, how could we write the example? Suppose, for instance, we wished to add 2 and 5 and multiply the result by 3. We could not write $2 + 5 \times 3$ for this would mean that only the 5 is to be multiplied by the 3 and then the 2 added. We write it this way:

$$(2 + 5)\,3 \text{ or better } 3(2 + 5) \text{ which equals } 3 \times 7 = 21$$

A *parenthesis* ( ) is used to show that the operations inside it are to be performed first, or that the quantities in it are to be treated as a whole.

Exercises

Find the value of the following:

1. $7 \times 3 + 4$
2. $8 - 2 \times 3$
3. $3 + 4 \div 2$
4. $5 \times 6 \div 3$
5. $8 \div 2 \cdot 2$
6. $4 + 5 \times 3 - 1$
7. $6 \times 2 - 4 \div 2$
8. $3 + 2 \times (3 + 4)$
9. $(2 + 5)(3 - 1)$
10. $18 \div 3 \div 3 + 4$

11. $2 + 2 \times 4 - 4 \div 2 \cdot 2 + 2 \times 4 \div 2 - 2$

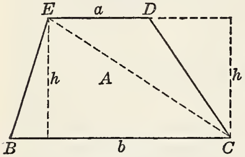
12. $8 - (2 + 3) + 3(5 - 1) - 6 \div 3 + 1$

13. $\frac{5(9 + 3)}{2}$ **14.** $\frac{6(2 + 8)}{5}$ **15.** $\frac{3(14 + 7)}{7}$ **16.** $\frac{4(15 + 10)}{10}$

17. In Exercises 15 and 16, would you get the same answer if you divided 7 into 7 and 10 into 10 before adding? Try it.

Parentheses in Formulas

The *trapezoid* is a four-sided figure, two of whose sides are parallel, that is, run in the same direction, as *ED* and *BC*. It can be cut into two triangles *EBC* and *EDC*. Do you think that these triangles have equal altitudes (heights)? What is the area of triangle *EBC* if *h* is the altitude and *b* is the base? Of *CDE* if *h* is the altitude and *a* the base?



Trapezoid

The area of a trapezoid equals half its altitude multiplied by the sum of the two parallel sides.

Formula: $A = \frac{1}{2} h(a + b)$

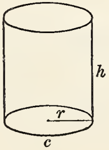
1. Find A if:

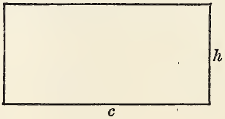
(*a*) $h = 10, a = 7, b = 11$ (*c*) $h = 9, a = 12, b = 14$

(*b*) $h = 8, a = 4\frac{1}{2}, b = 6\frac{1}{2}$ (*d*) $h = 7, a = 5, b = 8$

(*e*) If h, a, and b are all doubled, by what is A multiplied?

2. Here is the cylinder again. What is the shape of the top and bottom? If the radius is r, what is the area of the bottom? Of the top? Of both?





If this cylinder were made of paper and we cut it straight down from top to bottom and spread it flat, it would form a rectangle. What line of the cylinder would become the altitude of this rectangle? What line would become its base? Can you write a formula for finding c if r is known?

The formula for the surface of a cylinder can be put in the form:

$$S = 2\pi r(r + h)$$

Find the surface of a cylinder if $r = 10$ and $h = 4$.

3. The formula for the surface of a cone (page 40) is:

$$S = \pi r(r + s)$$

Find the surface of a cone if:

(*a*) $r = 7$ and $s = 9$

(*b*) $r = 14$ and $s = 9$

(*c*) $r = 7$ and $s = 18$

(*d*) $r = 14$ and $s = 18$

(*e*) Is the surface doubled when r is doubled? When s is doubled? When both are doubled?

The Formula in the Home

1. How many bushels will a bin 50 in. long, 40 in. wide, and 36 in. deep hold?

$$\text{Formula: } b = \frac{lwh}{2150}$$

2. How many cords of wood are there in a pile 18 ft. long, 4 ft. wide, and 8 ft. high?

$$\text{Formula: } c = \frac{lwh}{128}$$

3. How many tons of coal will a bin 12 ft. long, 8 ft. wide, and 3 ft. high hold?

$$\text{Formula: } t = \frac{lwh}{36}$$

4. How many square yards of canvas are needed to make a wigwam in the shape of a cone for Robert if the slant height is to be 6 ft. and the radius 3 ft.?

$$\text{Formula: } y = \frac{\pi rs}{9}$$

5. How many ice cream cones $3\frac{1}{2}$ in. high with a 1 in. radius can be filled level full from a gal. of ice cream?

$$\text{Formula: } n = \frac{693}{\pi r^2 h}$$

6. Are you afraid of lightning? Would you like to know how far away from you it is? In the next thunder storm, hold a watch to your ear, and when you see a flash, count the ticks of the watch until you hear the crash. If t is the number of ticks, the distance in miles is:

$$\text{Formula: } d = \frac{t}{25}$$

How far away is the lightning if the watch ticks 50 times between the flash and the sound? 60 times? 15 times?

7. The gable end of a house is 30 ft. wide and 20 ft. high. How many square feet of boards are needed to cover it?

$$\text{Formula: } n = \frac{wh}{2}$$

8. The grocer offers Mrs. Brown two cans of peaches for the same price. One has a radius of $1\frac{1}{2}$ in. and a height of 4 in. The other has a radius of 2 in. and a height of 3 in. Which can holds more?

$$\text{Formula: } V = \pi r^2 h$$

The Formula in Science and Engineering

1. The piston of a steam engine is 10 in. in radius. If there is a steam pressure of 90 lbs. per sq. in., what is the total pressure on the piston?

$$\text{Formula: } p = 90\,\pi r^2$$

2. How many times does a bicycle wheel, whose radius is 14 in., turn around in going 1 mi.?

$$\text{Formula: } t = \frac{5280 \cdot 12}{2\,\pi r}$$

3. A formula for force is $F = ma$. Find F if $m = 350$ and $a = 12$.

4. The horse-power of an engine is given by the formula

$H = \frac{PLAN}{550}$. Find H when $P = 90$, $L = 25$, $A = 330$, and $N = 5$.

5. South America has the general shape of a triangle whose base is 5000 mi. and whose height is 3000 mi. What is its area in square miles?

$$\text{Formula: } A = \tfrac{1}{2}hb$$

6. A gasoline tank is a cylinder. If its height is 11 ft. and its radius is 4 ft., how many gallons will it hold?

$$\text{Formula: } g = \frac{1728\,\pi r^2 h}{231}$$

The Formula in Business

1. In business, to find the capital of a company, we subtract the liabilities from the assets.

$$\text{Formula: } C = A - L$$

(*a*) Find C when:

$A = 8{,}000$ and $L = 5000$ $\quad$ $A = 8000$ and $L = 7000$

$A = 11{,}000$ and $L = 5000$ $\quad$ $A = 9000$ and $L = 6000$

(*b*) Does C become larger or smaller if:

A is increased, but L remains the same?

A remains the same, but L is increased?

Both are increased by the same amount?

(*c*) Brooks and Parker have assets of $28,500 and liabilities of $17,600. What is their capital?

Percentage is found by multiplying the base by the rate.

$$\text{Formulas: } p = br, \quad b = \frac{p}{r}, \quad r = \frac{p}{b}$$

2. (*a*) Using the proper formula, fill the blanks in this table:

b	r	p
$500	.05	
$300		$60
	.20	$80
$400		$50
	.30	$30

(*b*) A merchant sells \$800 worth of goods (b) at a profit of \$100 (p). What rate (r) of profit does he make on the sale?

(*c*) What profit (p) must I make on a sale of goods worth \$900 (b) to make 20% ($r = .20$)?

(*d*) Mr. Smith has a house assessed at \$14,000 (b). His tax bill is \$560 (p). What is the tax rate?

(*e*) A teacher has a salary of \$2000 (b), but he pays 5% of it (r) into a pension fund. What amount is deducted from his yearly salary?

(*f*) A clothier gained \$7.50 on a sale. This was 15% of the cost. What was the cost?

(*g*) What premium should I pay for insuring my house for \$3000 if the rate is \$.30 per \$100?

(*h*) 60% of what number is 18?

(*i*) 6 oz. is what per cent of a pound?

(*j*) A broker makes a commission of 3%. What amount must he sell per year to make a salary of \$2400?

(*k*) What per cent of 20 is 3?

(*l*) What per cent of 2 yds. is 2 ft.?

(*m*) Milk contains 4% butter fat. How much milk is needed to obtain 5 lbs. of butter fat?

3. The parcel post charge in cents for sending a package 800 mi. is $c = 9 + 6(p - 1)$, where p is the number of pounds.

(*a*) Find the cost of sending a package weighing 8 lbs. 16 lbs.

(*b*) Does it cost twice as much to send a 16-lb. package as to send an 8-lb. one?

(*c*) If you were sending goods by parcel post to one person, would you make the packages large or small? Give your reason.

4. In a certain state a married man can find the amount of his income tax by using the formula $T = \dfrac{i - 2500 - 400\,c}{50}$ where T is the amount of the tax, i his income, and c the number of children he has. What tax is paid by a married man whose income is:

(*a*) \$3000, if he has no children? If he has 1 child?

(*b*) \$6000, if he has no children? If he has 2 children?

(*c*) Does a man earning $6000 pay twice as much tax as one earning $3000?

(*d*) If a man's income increases, does his tax increase? If his income is doubled, is his tax doubled, more than doubled, or less than doubled?

5. In business, profit is often figured on the selling price. The merchant wishes to sell the goods at such a price that the profit will be a certain per cent of that price. He can determine this easily by using the formula $s = \frac{c}{1 - p}$ where c is the cost, p the per cent of profit expressed as a decimal, and s the selling price. If the profit, for example, is 25%, then $p = .25$.

Find the selling price if:

(*a*) The cost is $100 and the profit 20% of the selling price

(*b*) The cost is $27 and the profit 25% of the selling price

(*c*) The cost is $2.43 and the profit 10% of the selling price

(*d*) If c remains constant, what effect on s does increasing p have?

(*e*) Why should the merchant want to make the profit on each article small rather than large? What possible advantage might a smaller profit on each article be to him?

6. A merchant wishes to mark up goods so that he can give the customer a discount from the marked price and still make a certain per cent of the selling price. If c is the cost, m the marked price, d the per cent of discount, and p the per cent of profit (a per cent such as 30% being expressed as the decimal .30), he can find the marked price by using the formula:

$$m = \frac{c}{(1 - p)(1 - d)}$$

Illustration. A clothier bought a suit for $9. At what price should he mark it so he can give the customer a discount of 40% of the marked price and still make a profit of 25% of the selling price?

Here $c = \$9$, $p = .25$, and $d = .40$

$$m = \frac{9}{(1 - .25)(1 - .40)} = \frac{9}{.75 \times .60} = 20$$

He should mark the suit $20.

Find the marked price for these:

Cost	Profit on Selling Price	Discount on Marked Price
(*a*) $8.40	20%	25%
(*b*) $13.50	25%	40%
(*c*) $14.00	30%	50%
(*d*) $26.00	20%	35%

7. When the profit is figured on the cost instead of the selling price, the formula becomes:

$$m = \frac{c(1 + p)}{1 - d}$$

Find the marked price for each of these:

Cost	Profit on Cost	Discount on Marked Price
(*a*) $14.00	20%	30%
(*b*) $19.50	30%	35%
(*c*) $6.60	25%	40%
(*d*) $7.50	30%	25%

8. The formula for finding the amount at compound interest is $A = p(1 + r)^n$ where A is the amount, p the principal, r the rate, and n the number of years if the interest is compounded annually.

(*a*) Find the amount of $100 at 4% compounded annually for 2 yrs.

(*b*) Find the amount of $500 at 3% compounded annually for 3 yrs.

Fun from Formulas

1. James asked, "If I saved 1¢ on Jan. 1st, 2¢ on Jan. 2nd, 3¢ on Jan. 3rd, and so on up to 365¢ on Dec. 31, how much money would I have saved in the year?" Henry found the answer in 1 min. Could you beat that? He used the formula $s = \frac{n(f + l)}{2}$ where n is the number of days, f the first amount saved, and l the last amount saved. Find the answer.

2. Clara said, "Think of a number. Multiply it by 4. Add 8. Divide by 2. Subtract 1. What is your answer?" "15," answered Josie. "You started with 6," said Clara. She used the formula $n = \frac{a - 3}{2}$ where a is the answer she received and n the number taken. Clara made this formula herself. See if you can make a formula for a similar example. Mollie's answer was 25. What number did she start with?

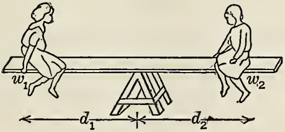
3. Sadie would not tell anyone her age, but she did not know algebra. Tom said, "Multiply your age by 3. Add 7. Now subtract your age. Now add 5. What is your answer?" Sadie replied, "46." Tom found her age by using the formula $a = \frac{n - 12}{2}$ where a is her age and n is the number she answered. What was Sadie's age?

4. Emily said to Helen, "Think of a number. Multiply it by 5. Add 6. Multiply by 4. Add 9. Now multiply by 5. What is your answer?" Helen replied that it was 565. Emily told her that she started with ——. No, we'll let you figure it out for yourself. Emily used the formula $n = \frac{a - 165}{100}$. With what number did Helen start?

Katherine's answer was 265. With what number did she start?

THE USE OF SUBSCRIPTS IN FORMULAS

Two children are riding on a teeter board or seesaw. If their weights are 40 lbs. and 60 lbs., how must their distances from the point of support, the fulcrum, compare so that the board will balance? Should the heavier or lighter child sit farther from the fulcrum?



The scientist calls a board supported in this way a *lever,* and he finds that it will balance when the weight of one child multiplied by his distance from the fulcrum equals the weight of the other child multiplied by his distance from the same point.

Here we have two different weights and two different distances. How shall we represent them? Of course we could use w for one weight and x for the other, but it is more convenient to use only w for a weight. However, the same w cannot represent two different numbers. Therefore, we put a small number called a ***subscript*** after the w and slightly below it, and we shall understand that w_1 (read w sub-one) is just as different from w_2 (w sub-two) as h is from t. Our formula then becomes:

$$w_1d_1 = w_2d_2 \text{ or } w_1 = \frac{w_2d_2}{d_1}$$

Exercises

1. (*a*) John, who weighs 60 lbs. is 4 ft. from the fulcrum of a teeter board. What is Robert's weight if he is 6 ft. from that point? ($w_2 = 60$, $d_2 = 4$, $d_1 = 6$. Find w_1.)

(*b*) Find w_1 if $w_2 = 10$, $d_1 = 8$, and $d_2 = 12$.

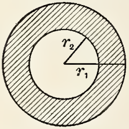
(*c*) Find w_1 if $w_2 = 10$, $d_1 = 8$, and $d_2 = 24$.

(*d*) How does doubling d_2 affect w_1?

(*e*) Find w_1 if $w_2 = 10$, $d_1 = 16$, and $d_2 = 12$.

(*f*) How does doubling d_1 affect w_1?

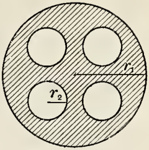
2. When a ball is thrown down from a cliff, its velocity (speed) can be found by using the formula $V = v_1 + v_2t$. Find V if $v_1 = 50$, $v_2 = 32$, and $t = 10$.



3. The distance from a curved mirror to the image is given by the formula $d_1 = \frac{d_2r}{2\,d_2 - r}$. Find d_1 when $r = 6$ and $d_2 = 5$.

4. The area of a washer, the shaded part in the figure, is given by the formula $A = \pi(r_1^2 - r_2^2)$. Find the area when $r_1 = 3$ and $r_2 = 2$.

5. A formula used by electrical engineers is $E = C(R_1 + NR_2)$. Find the value of E when $C = 2$, $N = 15$, $R_1 = 25$, and $R_2 = 2$. When $C = .25$, $N = 10$, $R_1 = 1.8$, and $R_2 = .42$.



6. The force needed to change the velocity of a body is $f = \frac{wv_2 - wv_1}{t_2 - t_1}$. Find f if $w = 100$, $v_1 = 12$, $v_2 = 44$, $t_2 = 6$, and $t_1 = 2$.

7. The part of a circular disc of radius r_1 left after 4 holes of radius r_2 have been cut in it is given by the formula $A = \pi(r_1 + 2r_2)(r_1 - 2r_2)$. Find A when $r_1 = 10$ and $r_2 = 2$.

MAKING FORMULAS FROM RULES

Class Exercises

Make a formula using the initial letters of the important words:

1. The Fahrenheit reading on a thermometer is $\frac{9}{5}$ of that on a centigrade thermometer plus 32.

2. The surface of a cube is 6 times the square of an edge.

3. The amount of money at interest is found by multiplying the principal, rate, and time and adding the product to the principal.

4. The volume of a pyramid is $\frac{1}{3}$ the product of its base and height.

5. The height of a triangle is found by dividing twice its area by its base.

6. To find the perimeter of a rectangle, add twice its length to twice its width.

7. The sum of money at interest can be found by dividing the interest by the product of the rate and time.

8. The assets of a firm equal the sum of its liabilities and capital.

9. The selling price is found (*a*) by subtracting the loss from the cost. (*b*) By adding the gain to the cost.

10. The commission received by a broker is the product of the selling price and the rate of commission.

Optional Exercises

11. The volume of a sphere equals $\frac{4}{3}\pi$ times the cube of the radius.

12. The length of a rectangular box is found by dividing the volume by the product of the width and height.

13. The area of a trapezoid equals $\frac{1}{2}$ its height multiplied by the sum of the top and bottom.

14. The number of edges of a solid figure equals the sum of its faces and vertices decreased by 2.

15. The cost of sending a package is 5¢ more than 4 times the weight in lbs.

16. When profit is figured on the selling price, the selling price of an article can be found by dividing the cost by 1 minus the per cent of profit.

17. The amount of illumination on your book when you read at night is found by dividing the candle power of the light (c) by the square of the distance (d) from the light to your book.

Honor Work

18. The height of a trapezoid can be found by dividing twice the area by the sum of the two bases, b_1 and b_2.

19. When profit is figured on the selling price, you can determine the marked price by subtracting the per cents of discount and profit from 1, then adding the product of these per cents, and finally dividing this result into the cost.

20. To find the total surface of a cylinder, multiply the radius by the height, add this result to the square of the radius, and then multiply this sum by 2π.

21. A married man can find his income tax by deducting \$2500 and the product of \$400 and the number of his children from his salary and then multiplying the result by 1%.

Making a table.

Illustration. Miss Strong wrote $x + y = 12$ on the board and asked, "What are the values of x and y?" Dorothy replied, "$x = 6$ and $y = 6$"; Paul said, "$x = 8$ and $y = 4$"; John said, "$x = 1$ and $y = 11$"; and Robert said, "$x = 5$ and $y = 7$." Which of them is right, or are they all right? Can you find still other values of x and y that make $x + y = 12$? Could x be a fraction?

It is evident that there is no single "answer" to this equation. The value of y depends on the value of x. When x changes, y will change.

Complete this table by finding the values of y for the values of x given:

x	1	2	3	4	5	6	7	8	9	10	11
y	11				7	6		4			

Class Exercises

1. Mr. George earns $.40 an hr. Complete the table showing his earnings for the hours given.

Hours...	1	2	3	4	5	6	7	8	9	10
Wages...	.40	.80								

2. An automobile travels 20 mi. an hr. Complete the table showing the distance it goes in the hours given:

Hours....	1	2	3	4	5	6	7	8	9
Distance..	20		60						

3. Beef costs $.30 a lb. Complete this table showing the cost of the number of pounds given:

Number of pounds...	1	2	3	4	5	6	7	8	9
Cost...............	.30			1.20					

4. The circumference of a circle is $\frac{22}{7}$ times the radius.

r	0	7	14	21	28	35	42	49
C	0	22				110		

Make a table of values for:

5. $p = 4s$ from $s = 1$ to $s = 6$

6. $c = 6w$ from $w = 1$ to $w = 6$

7. $x + y = 9$ from $x = 1$ to $x = 9$
8. $i = .5\,t$ for even values of t from $t = 2$ to $t = 10$
9. $d = 5\,t + 3$ from $t = 0$ to $t = 5$
10. $v = 32\,t$ from $t = 0$ to $t = 6$
11. $h + w = 14$ from $w = 1$ to $w = 13$
12. $V = 20\,h$ from $h = 0$ to $h = 5$
13. $a + b = 12$ from $a = 0$ to $a = 12$
14. $c = \frac{22}{7}\,d$ from $d = 0$ to $d = 7$

Optional Exercises

15. $n = 12 - d$ for even values from $d = 0$ to $d = 10$
16. $d = 1.4\,s$ from $s = 1$ to $s = 6$
17. $V = 10\,s^2$ from $s = 0$ to $s = 5$
18. $A = \frac{22}{7}\,r^2$ from $r = 2$ to $r = 7$
19. $V = e^3$ from $e = 0$ to $e = 5$
20. $s = 16\,t^2$ from $t = 0$ to $t = 5$

Honor Work

21. $K = e^2 + 40\,e$ from $e = 0$ to $e = 5$
22. $R = .001(t + 273)^2$ for $t = 0$ and $t = 27$
23. $PV = 840$ from $P = 1$ to $P = 8$
24. $L = 100(1 + .03\,t + .0001\,t^2)$ from $t = 0$ to $t = 40$ by 10's

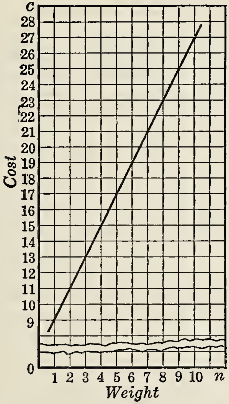
THE FORMULA GRAPH

John Hunter got a position at the Post Office just before Christmas. He wanted to be able to tell customers quickly the cost of sending parcels. He found that for the third zone the rate was 9¢ for the first pound and 2¢ for each additional pound. John wrote the formula $c = 9 + 2(n - 1)$ and then he changed it to $c = 2\,n + 7$. Can you see how he got this result?

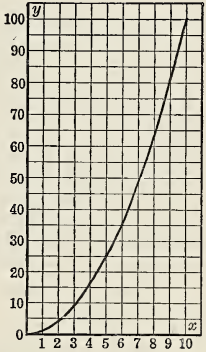
Using this formula, he next made the table:

n	1	2	3	4	5	6	7	8	9	10
c	9	11	13	15	17	19	21	23	25	27

Finally he decided to make a large graph so he could read results directly from it. So he laid off the values of n at equal intervals on the horizontal axis and selected a scale for c for the vertical axis. Then he marked points. For $n = 1$, $c = 9$, he put a point directly above the 1 on the n axis and at a height of 9. Above 2 he put a point at a height of 11 and so on. Finally drawing a line through all his points, he obtained this graph.



JOHN'S PARCEL-POST GRAPH



ENGINEER'S SQUARE-ROOT GRAPH

Engineers use graphs for many purposes. Here is one from which they can find squares and square roots of numbers.

Let $y = x^2$. From this we can make the table:

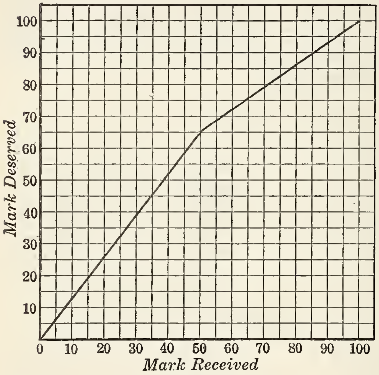
x	1	2	3	4	5	6	7	8	9	10
y	1	4	9	16	25	36	49	64	81	100

Laying off x on the horizontal axis and y on the vertical axis we get this graph.

From the graph estimate the squares of 3.5, of 6.3, of 4.8. The square roots of 30, of 54, of 20, of 43.

In making graphs, we generally follow these rules:

When the formula is solved for one letter, that is, when one letter is alone on the left side of the equation, we use that on the vertical axis.



When one of the quantities is time, we measure that off on the horizontal axis.

Mr. Wright found that the test he gave his class was too hard so he decided to mark a pupil who got 50% up to 65%, but to leave 0 and 100 unchanged. So he drew this graph, making the horizontal axis the mark the pupil received and the vertical axis the mark the pupil deserved.

What mark should he give a pupil who made 80%? 40%? 65%? 50%? 90%? 55%?

Class Exercises

Draw a graph for each of these exercises.

1. $i = 4t$
2. $A = 10h$
3. $v = 32t$
4. $c = \frac{44}{7}r$
5. $c = 2\frac{1}{2}i$
6. $h = d + 5$
7. $c = 2n + 6$
8. $c = 20m + 10$
9. $A = 100 + 6t$

Optional Exercises

10. $F = \frac{9}{5}C + 32$
11. $d = 16t^2$
12. $A = e^2$
13. $A = \frac{22}{7}r^2$
14. $V = 22r^2$
15. $A = 6e^2$

Honor Work

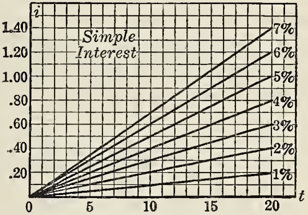
16. $d = 20t + 16t^2$ **17.** $d = 50t - 6t^2$ **18.** $T = 22r + \frac{22}{7}r^2$

Thought Questions

Which of these formulas have straight-line graphs? Which have curves? Can you tell in advance from the equation whether its graph will be straight or curved?

The Graph in Business

Simple interest graphs. The graph shown here represents the interest on $1 for t yrs. at different rates. Using this graph, estimate the interest on $1 for:



1. 10 yrs. at 6%, at 4%, at 7%.

2. 12 yrs. at 2%, at 5%, at 4%.

3. At 4% how many years will it take $1 to gain $.60? $.48? $.80?

4. How much more interest will $1 earn in 8 yrs. at 7% than at 4%? At 6% than at 2%?

5. In how many years will $1 earn $.40 at 6%? At 7%? At 5%?

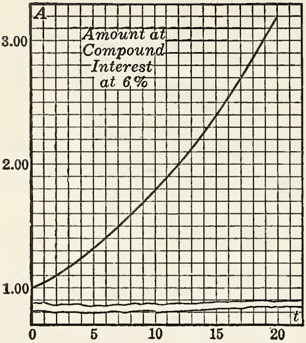
6. How long will it take $1 to earn $.30 more at 5% than at 2%?

7. How can you use this graph to find the interest on $100? Find the interest on $100 for 20 yrs. at 1%. At 3%.

8. What change should you make in the scale on this graph so that it would give the amount of $1 instead of the interest on $1?

9. Using the graph find the amount of $1 for 15 yrs. at 4%. For 10 yrs. at 6%.

Compound-interest graphs. The following graph shows the amount of $1 at 6% compounded annually for t yrs.



1. At 6% compounded annually, what will \$1 amount to in 5 yrs.? In 12 yrs.? In 20 yrs.?

2. At 6% compounded annually, in how many years will \$1 amount to \$2? To \$3? To \$2.50? To \$1.80?

3. By how much does the interest for the second 10 yrs. exceed that for the first 10 yrs.?

4. At 6% compounded annually, in how many years will a sum of money double itself? Triple itself?

5. How should you change the scale on this graph so that you could read the interest on \$1 instead of the amount of \$1? Find the interest on \$1 for 10 yrs.

Class Exercises on Formulas: Review

Find the value of:

1. $V = \frac{1}{3}bh$ when $b = 16$ and $h = 12$

2. $s = \frac{1}{2}gt^2$ when $g = 32$ and $t = 10$

3. $i = prt$ when $p = 300$, $r = .04$, and $t = 7$

4. $A = p + prt$ when $p = 650$, $r = .05$, and $t = 4$

5. $K = \dfrac{h(t + b)}{2}$ when $h = 11$, $t = 9$, and $b = 15$

6. $C = \frac{5}{9}(F - 32)$ when $F = 95$; when $F = 32$

7. $A = \pi r^2$ when $\pi = 3.14$ and $r = 10$

8. $w_1 = \dfrac{w_2 d_2}{d_1}$ when $w_2 = 80$, $d_1 = 6$ and $d_2 = 9$

Optional Exercises on Formulas: Review

9. $S = \dfrac{rl - a}{r - 1}$ when $r = 3$, $a = 5$, and $l = 1035$

10. $T = 2\pi r(r + h)$ when $\pi = \frac{22}{7}$, $r = 14$, and $h = 9$

11. $C = \dfrac{En}{R + nr}$ when $E = 1.5$, $n = 45$, $R = 90$, and $r = 2$

12. $P = \frac{A}{1 + rt}$ when $A = 484$, $r = .03$, and $t = 7$

13. $E = \frac{Re + re}{r}$ when $e = 3$, $R = 80$, and $r = 5$

14. $m = \frac{c(1 + p)}{1 - d}$ when $c = 6.60$, $p = 25\%$, and $d = 40\%$

15. $n = \frac{693}{\pi r^2 h}$ when $\pi = 3.1$, $r = 1.5$, and $h = 3$

16. $d = k + vt + 16\,t^2$ when $k = 60$, $v = 55$, and $t = 4$

Honor Work on Formulas: Review

17. $\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$. Find f when $p = 8$ and $q = 12$.

18. $p_1v_1 = p_2v_2$. Find v_1 when $p_1 = 15$, $p_2 = 80$, and $v_2 = 120$.

19. $m = \frac{c}{1 - d - p + dp}$ when $c = 13.60$, $d = 20\%$, and $p = 15\%$

20. $T = \pi r^2 + \pi rh$. Find h when $T = 471$, $\pi = 3.14$, and $r = 10$.

21. $H = \frac{(w_1 + w_2)(t_2 - t_1)r - (x + y + z)}{.55\,w}$ when $w_1 = 50$, $w_2 = 30$, $w = 20$, $t_2 = 87$, $t_1 = 62$, $r = 11$, $x = 22$, $y = 34$, and $z = 21$

22. $\left(\frac{P + a}{V^2}\right)(V - b) = k$. Find k when $P = 80$, $V = 10$, $a = .01$, and $b = .05$.

Review of Formula Graphs

1. (*a*) Draw the graph of $V = 4\,h$ from $h = 0$ to $h = 6$.

(*b*) By extending your graph, find the value of V when $h = 7$.

(*c*) When h increases, does V increase or decrease?

(*d*) When h becomes 3 times as large, does V become exactly 3 times as large as before?

2. (*a*) Draw the graph of $A = 200 + 12\,t$ from $t = 0$ to $t = 5$.

(*b*) By extending your graph find A when $t = 6$.

(*c*) From your graph find t when $A = 242$.

(*d*) When t increases, does A increase or decrease?

(*e*) When the value of t is doubled, is the value of A doubled?

3. (*a*) Draw the graph of $V = \dfrac{12}{P}$ from $P = 1$ to $P = 12$.

(*b*) From your graph find the value of V when $P = 9$.

(*c*) From your graph find P when $V = 5$.

(*d*) When P increases, does V increase or decrease?

(*e*) When P is doubled, what happens to the value of V?

(*f*) Can you find the value of V when $P = 0$?

4. (*a*) Draw the graph of $C = \frac{5}{9}(F - 32)$ from $F = 32$ to $F = 95$.

(*b*) From your graph find C when $F = 68$.

(*c*) When F increases, does C increase or decrease?

(*d*) Is C doubled when F is doubled?

5. (*a*) Draw the graph of $A = s^2$ from $s = 0$ to $s = 10$.

(*b*) From the graph find A when $s = 6.5$. Find s when $A = 20$.

(*c*) When s increases, does A increase or decrease?

(*d*) When s becomes 3 times as large, by what number is A multiplied?

(*e*) Which is increasing the faster, s or A?

Test in Evaluating Formulas

Formula	*Given Values*	*Answer*
1. $p = a + b + c$	$a = 12, b = 8, c = 9$	$p =$ . . . .
2. $A = bh$	$b = 14, h = 10$	$A =$ . . . .
3. $A = \frac{1}{2}h(b + t)$	$h = 15, b = 13, t = 7$	$A =$ . . . .
4. $A = P + Prt$	$P = 500, r = .04, t = 6$	$A =$ . . . .
5. $E = \frac{1}{2}mv^2$	$m = 14, v = 9$	$E =$ . . . .
6. $T = \pi r^2 + \pi rh$	$\pi = \frac{22}{7}, r = 7, h = 12$	$T =$ . . . .
7. $s = \dfrac{n - k}{p - k}$	$n = 26, k = 8, p = 14$	$s =$ . . . .
8. $V = \dfrac{vm_1}{m_2 - m_1}$	$v = 50, m_1 = 12, m_2 = 17$	$V =$ . . . .
9. $h = \dfrac{s - \pi r^2}{\pi(r + 3)}$	$\pi = \frac{22}{7}, r = 5, s = 88$	$h =$ . . . .
10. $w = \dfrac{800\,kn^3}{r^2 + 40\,n^2}$	$k = 6.6, n = 3, r = 6$	$w =$ . . . .

Test in Making Formulas from Rules

Make formulas using initial letters of the important words:

1. The perimeter of a regular hexagon is 6 times an edge.

2. The amount of money at interest equals the sum of the principal and interest.

3. The sum of a certain group of numbers is $\frac{1}{2}$ the number of them multiplied by the sum of the first and last numbers.

4. When 2 boys balance on a seesaw, the weight of 1 equals the product of the weight of the other and his distance from the fulcrum divided by the distance of the first from the fulcrum.

5. A married man's income tax is found by taking 1% of the remainder after deducting $2500 and $400 times the number of his children from his income.

Matching Test in Recognizing Formulas

Here are important formulas that you should recognize. Write the numbers 1 to 14 in a column and after each number, write the letter of the formula that corresponds to the statement after the number.

The formula for finding the:

1. Area of a rectangle
2. Perimeter of a triangle
3. Volume of a box
4. Area of a circle
5. Amount at simple interest
6. Volume of a cube
7. Circumference of a circle
8. Percentage
9. Area of a triangle
10. Perimeter of a rectangle
11. Area of a trapezoid
12. Weight on a seesaw
13. Distance a stone will fall
14. Capital of a corporation

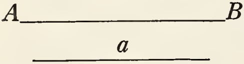
a. $w_1 = \frac{w_2 d_2}{d_1}$
b. $V = lwh$
c. $A = P + Prt$
d. $C = A - L$
e. $d = \frac{1}{2} gt^2$
f. $A = lw$
g. $p = a + b + c$
h. $p = 2\,l + 2\,w$
i. $A = \pi r^2$
j. $V = e^3$
k. $A = \frac{1}{2} hb$
l. $p = br$
m. $c = 2\,\pi r$
n. $A = \frac{1}{2} h(b_1 + b_2)$

Chapter 3

GEOMETRIC MEASUREMENT

MR. Deems, a surveyor, says that he ran a line at an angle of 87° with the road. What did he mean? If you wish to understand the language of surveyors, or if you are interested in astronomy, architecture, or any kind of engineering, you need some knowledge of angles and lines.

You know what a straight line is, but do you know that a mathematician thinks of that line as having no end but going on and on? The part from A to B he calls a ***line segment.*** Do you know that a line has no width or thickness, but only length? It is like the upper edge of our line AB.



We read a line segment by naming the two capital letters at its ends, as AB, or by reading one small letter placed somewhere near its middle, as line segment a. When the small letter is used, it generally stands for the number of units in the length of the segment. For example, a could stand for 7 in. or 16 ft. When no confusion can result, we generally call either a line or a line segment simply a line.

Exercises

1. Draw a line on paper. Is it a line or a line segment? Explain.

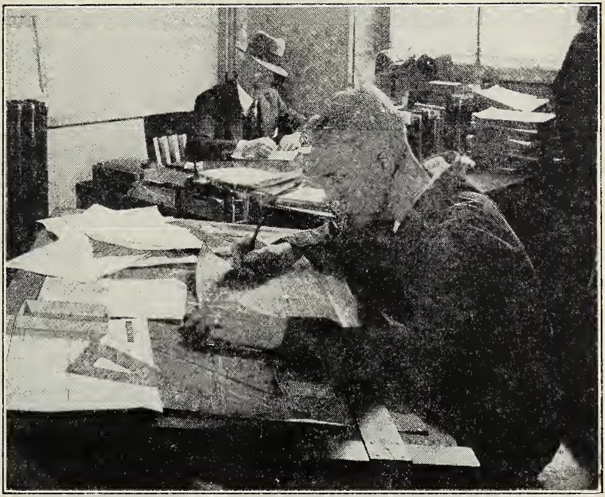
2. Is it a true line segment? Has it width? Thickness?

3. Which do you think better represents a line segment, a pencil, a piece of wire, or a piece of spider web? Why?

4. If the end of a line segment is a point, how large is a point?

5. Make a point on your paper, and draw a straight line through it. Can you draw other straight lines through that same point?

6. Make two points on your paper, and draw a straight line through both of them. Can you draw other straight lines that pass through both points?



Photograph from Philip D. Gendreau, N.Y

THE ENGINEER AT WORK

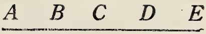
Drafting plans for large buildings requires a thorough knowledge of mathematics.

7. How many of its points must you know to be able to tell just where a straight line goes?

We say that two points *determine* a straight line because only one such line can pass through both of them, and we can tell which line is meant when two of its points are named.

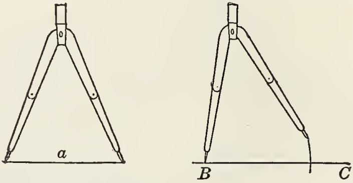
8. Point out the segment AB, BC, DE.

9. What segment equals $AB + BC$? $CD + DE$? $AB + BC + CD$?



10. What segment equals $AC - AB$? $AE - AB - DE$? $AD - CD$?

How to lay off a line segment. To lay off a length on BC equal to segment a, open your compasses until the points will rest on the ends of a. Then putting a point of the compasses on B, make a mark across BC.



When we copy a figure accurately, using only compasses and ruler, we say that we *construct* it. If we also use other drawing instruments, of which you will learn later, we say that we *draw* the figure.

Exercises

1. Draw a segment, and mark it a. Now construct a segment 3 times as long as a.

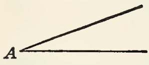
2. Draw two segments k and n. Construct a segment equal to their sum.

3. Draw three segments r, s, and t. Construct $r + s + t$.

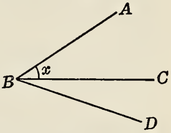
4. Draw two segments a and b. Construct $2a + b$, $2a + 3b$, $2a - b$, $2a + 2b$, $3a - 2b$.

The angle. An angle is formed by two straight lines that meet at a point. The lines are the ***sides*** of the angle, and the point is its ***vertex.***

We generally read an angle by a capital letter near its vertex, as angle A (written $\angle A$). However when two or more angles have the same vertex, we cannot tell by this method which of them is meant. We then read three letters, one at the vertex and one on each side, the letter at the vertex being read between the other two, as $\angle ABC$

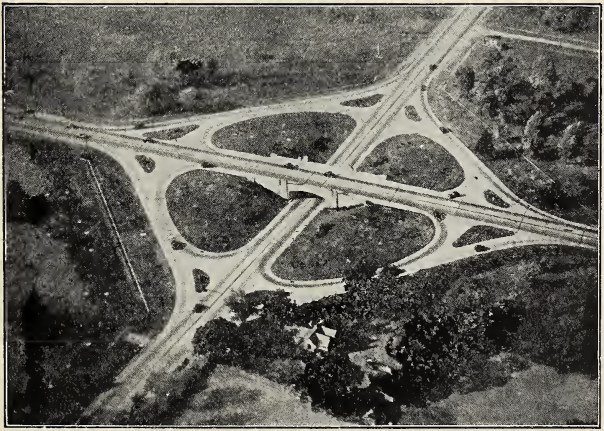


or $\angle CBA$. The letters are read in the order you would come to them if you drew the angle with one continuous stroke of your pencil. We can also read an angle by a small letter inside the angle. $\angle ABC$ can be read $\angle x$.



We may think of $\angle ABC$ as having been formed by turning a line around point B as a pivot from the initial position BC to the final position BA. The size of the angle depends on the amount of this turning and not on the length of the sides.

Illustration. Lay your book closed on the desk. Now open the front cover without moving the book. Notice that the cover turns around the back edge as the book opens. The size of the angle that the lower edge of the cover makes with the lower edge of the first page depends on the amount the cover has turned.



Courtesy of Missouri State Highway Dept.

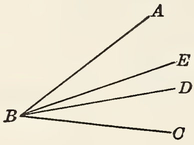
MODERN CROSSROADS BECOME GEOMETRICAL

Nowadays superhighways are made more safe for drivers by elimination of intersections: In the picture above 45° angles are used to solve the problem. In others complete circles are utilized.

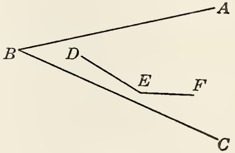
Exercises

1. Draw lines *AB* and *CD* crossing at point *E*. Read the angles formed by these lines.

2. There are six angles in this figure. Can you name all of them?

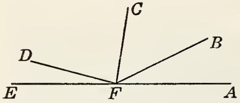


Ex. 2



Ex. 3

3. Which of the angles *ABC* and *DEF* is the larger? If you lengthen the lines *BA* and *BC*, does ∠*ABC* become larger?



4. Which is larger, ∠*AFB* or ∠*AFC*?

5. If ∠*CFD* is added to ∠*AFC*, what angle is the result?

6. If ∠*AFB* is subtracted from ∠*AFD*, what angle is the result?

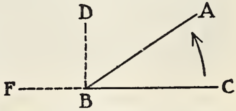
7. Is there any angle in the figure larger than ∠*AFD*?

8. If ∠*DFE* is added to ∠*AFD*, what angle is the result?

9. Draw a straight line *AB*. Now draw another straight line *CD* meeting *AB* at *D* so that 2 equal angles are formed at *D*. What is the name of this kind of angle?

CLASSIFICATION OF ANGLES

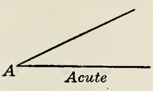
When the line *AB* has rotated around *B* until it takes the position *BF* in a straight line with *CB*, the angle formed is a ***straight angle*** (st. ∠). ∠*CBF* is a straight angle.

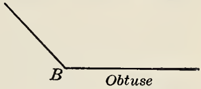


When line *AB* had turned half that amount to the position *BD*, the angle it formed with *CB* was a right angle. A ***right angle*** (rt. ∠) is half a straight angle. ∠*CBD* is a right angle.

Two lines that form a right angle are ***perpendicular*** to each other. *DB* is perpendicular to *FC*.

An angle smaller than a right angle is an ***acute angle,*** as ∠*A*. An angle larger than a right angle but smaller than a straight angle is an ***obtuse angle;*** as ∠*B*.

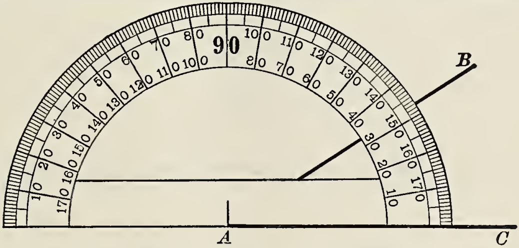




The degree. The ordinary unit for measuring angles is the degree (°) which is $\frac{1}{90}$ of a right angle. The straight angle then equals 180°, and a complete revolution about a point equals 360°. A degree may be divided into tenths, hundredths, etc., or into 60 equal parts called minutes (′).

60′ = 1°
90° = 1 rt. ∠
180° = 1 st. ∠

How to measure an angle. A ***protractor*** is an instrument for measuring the number of degrees in an angle. It is made in the form of a half circle divided into 180 equal parts called degrees. To find the number of degrees in an angle, we place the protractor so that its middle point is on the vertex *A* of the angle and so that the straight edge of it fits along one side *AC*. Then the point where the other side *AB* crosses the scale tells us the number of degrees in the angle. In the figure the angle is about 33°. Can you see how we found that answer?



The principal instruments used in engineering and navigation, such as the transit and sextant, are fundamentally protractors.

Exercises with the Protractor

1. Draw an acute angle *BAC*. Now using a protractor as shown in the figure, measure the number of degrees in your angle.

2. Draw an angle as near a right angle as you are able without using the protractor or compasses. Now measure it with a protractor, and note how many degrees it differs from a true right angle.

3. In the same way draw and afterwards measure angles of 45°, 135°, 30°, 60°, 120°, $22\frac{1}{2}$°, $67\frac{1}{2}$°, 20°, and 40°. In each case compare your estimate with the measured result.

4. Draw an angle whose sides are each 2 in. long, and measure it. Now lengthen the sides, making one of them 3 in. and the other 4 in. long. Measure the angle again. Has its size changed?

5. Draw two angles as nearly equal as you can estimate. Measure them, and determine the number of degrees by which they differ.

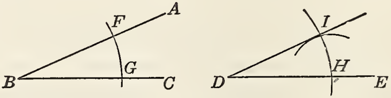
6. Draw an angle *ABC*. Now draw a line *BD* that you think will divide $\angle ABC$ into two equal parts. Measure the two parts, and determine their difference.

7. Draw an angle of 45°, and extend one of its sides through the vertex. Measure both angles. Find their sum.

8. Make the same drawing and measurements, starting with angles of 135°, 30°, 60°, 120°, 150°, $22\frac{1}{2}$°, and 40°. Could you have computed the number of degrees in each of the angles formed by extending the line, without measuring? How?

9. Draw two lines that cross each other. Measure two opposite angles. How do they compare in size?

How to copy an angle. To make an angle at *D* equal to $\angle B$:



1. Place the point of your compasses on *B*, and with any convenient opening make an arc cutting *AB* at *F* and *BC* at *G*.

2. Keeping the same opening of the compasses, put the point on *D*, and draw an arc cutting *DE* at *H*.

3. With your compasses measure *GF*. Then put the point on *H*, and make an arc cutting the other arc at *I*.

4. Using your ruler, draw *DI*. $\angle D$ will equal $\angle B$.

Exercises with Compasses and Ruler

Remember we say we construct a figure when we draw it accurately using compasses and straight-edge ruler only.

1. With your protractor draw an angle of 70°. Now construct an angle equal to it. Check with your protractor.

2. Draw the following angles, and construct an angle equal to each: (*a*) 15°, (*b*) 110°, (*c*) 90°, (*d*) 45°.

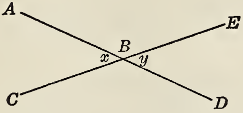
3. Draw an angle of 35°. Now construct an angle twice as large.

4. Draw any two angles, and then construct an angle equal to their sum.

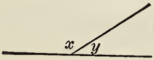
5. Draw an angle of 20°. Construct an angle (*a*) 3 times as large, (*b*) 4 times as large, (*c*) 6 times as large.

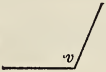
6. Draw three unequal angles, and construct an angle equal to their sum. Check with your protractor.

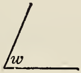
Vertical angles. When two straight lines cross, the opposite angles are called vertical angles, as $\angle x$ and $\angle y$.



Supplementary angles are two angles whose sum is a straight angle, as $\angle x$ and $\angle y$, or as $\angle v$ and $\angle w$. Each angle is the supplement of the other.



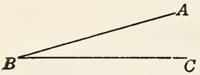




Class Exercises

1. If the line *AB* turns around point *B* in the direction opposite to that in which the hands of a clock move until it lies in a straight line with *BC*, name in order the four kinds of angles it forms with *BC*.

2. What kind of angle equals the sum of a right angle and an acute angle? What kind of angle equals their difference?



3. If an acute angle is subtracted from a right angle, what kind of angle is the result?

4. If an acute angle is subtracted from a straight angle, what kind of angle is the result?

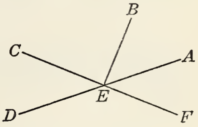
5. Compare an acute angle with its supplement.

6. If an acute angle grows larger, how does its supplement change? How large must the angle become to equal its supplement? To exceed its supplement?

7. What kind of angle is $\frac{2}{5}$ of a straight angle? $\frac{3}{5}$ of a straight angle?

8. Is $\frac{1}{2}$ an obtuse angle necessarily an acute angle? Is twice an acute angle necessarily an obtuse angle?

9. If 3 times an acute angle is an obtuse angle, what number of degrees must the acute angle exceed?



10. In this figure there are 6 pairs of supplementary angles. Can you name all of them? There are also 2 pairs of vertical angles. Name them.

11. What is the angle formed by the hands of a clock at 3 P.M.? At 6 P.M.? At 4 P.M.? At 2 P.M.? At 5 P.M.?

12. Through how many degrees does the minute hand of a clock turn in 1 min. of time?

13. How many degrees are there in the angle formed by the hands of a clock at 12:30 P.M.? At 2:30 P.M.?

14. Through how many degrees does the minute hand pass in 20 min.? In 13 min.?

15. If the earth turns completely around in 24 hrs., through how many degrees does it turn in 1 hr.? In 2 hrs.? In 15 min.? How long does it take the earth to turn through an angle of 60°? Of 1°?

Optional Exercises

16. Find the supplement of 50°. Of 67°18′.

17. Find the angle that is $\frac{2}{3}$ of its supplement.

18. If a part of a circle (an arc) has the same number of degrees as the angle at the center that cuts it off, how many degrees are there in a whole circle?

19. How many degrees are there in a half circle? In a quarter circle?

20. Using the formula $c = 2\pi r$, find the circumference of a circle whose radius is 7, 10, 1000, z.

21. What part of a circle is cut off by an angle at the center equal to: 90°? 60°? 45°? 20°? 57°? n°?

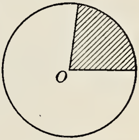
22. If the radius of the circle is 7, what is the length of an arc of 180°? 90°? 60°? n°?

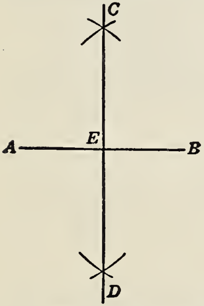
Honor Work

23. Using the formula $A = \pi r^2$, find the area of a circle whose radius is 7.

24. What part of the circle is shaded if $\angle O$ s: 90°? 60°? 100°? n°?

25. If $r = 7$, find the area of the shaded part (sector) when $\angle O = 90°$, when $\angle O = 60°$.



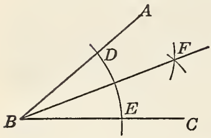


To *bisect* means to cut in two equal parts or in halves.

How to bisect a line segment. To bisect AB:

1. With A as center and with a radius more than half of AB, construct arcs on both sides of AB.
2. With B as center and with the same radius, construct arcs cutting the first arc at C and D.
3. With your ruler draw CD cutting AB at E.
4. E is the middle point of AB.

NOTE. CD is also perpendicular to AB. It is called the **perpendicular bisector** of AB.



How to bisect an angle. Draw an angle *ABC*. With *B* as center, draw an arc cutting *AB* at *D* and *BC* at *E*. With *D* and *E* as centers and with the same radius, draw two arcs crossing at *F*. With your ruler draw a straight line from *B* to *F*. *BF* bisects ∠*ABC*.

Construction Exercises

1. Draw a vertical line segment, and bisect it. Test with your compasses.

2. Draw a line segment. Then construct a segment $1\frac{1}{2}$ times as long.

3. Divide a line segment into 4 equal parts.

4. By the method learned above, can you divide a line segment into 3 equal parts? 5 equal parts? 8 equal parts? Construct those that are possible.

5. Bisect an angle of 45°, 60°, 120°. Test each angle either with your compasses or with your protractor.

6. Construct a 90° angle by bisecting a straight angle.

7. Draw an angle and divide it into 4 equal parts.

8. Can you divide an angle into 3 equal parts? 5 equal parts? 8 equal parts? Construct those that are possible.

9. Construct an angle of 45°, of 135°, of 22°30′.

10. Construct an angle $1\frac{1}{2}$ times a given angle.

11. At a point *C* on line *AB*, construct a line perpendicular to *AB*.

Thought Questions

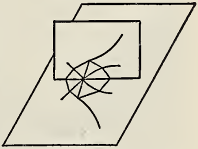
1. By the methods you have just learned, can you cut a line segment or an angle into n equal parts if n is 2? 3? 4? 5? 6? 8? 9? 16? 20? 24? 32? 64?

2. If an angle can be cut into n equal parts by these constructions, can a factor of n be 2? 3? 5? 7? Of what prime factors is n necessarily composed?

3. Can you divide an angle into $\frac{n}{2}$ equal parts if n is 1? If n is 3? Can n equal any whole number?

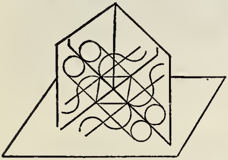
Symmetry and Beauty

1. Make a few heavy lines with ink on a sheet of paper, and fold it quickly before the ink dries. Has it become more beautiful?



2. Draw a few lines on paper, and hold a mirror perpendicular to the paper. Does the reflection add to the beauty of the figure?

3. If you can obtain 2 pieces of plane mirror, hold them perpendicular to the paper and perpendicular to each other. Does this still more increase the beauty of your diagram?



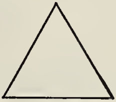
When a figure can be folded along a line so that the part on one side of the line will exactly fit the part on the other side, we say that it is *symmetric* with respect to that line. The line is called an *axis of symmetry.*

4. Look at the picture in the front of this book. Is it beautiful? Can you find an axis of symmetry in it?

5. How many axes of symmetry can you find in:
- (*a*) A square?
- (*b*) A rectangle?
- (*c*) An equilateral triangle?
- (*d*) An isosceles triangle?
- (*e*) A regular 6-sided figure?
- (*f*) A circle?

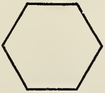
Square

Rectangle

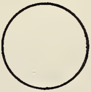


Equilateral Triangle

Isosceles Triangle

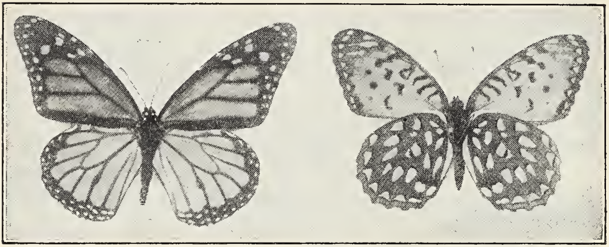


Regular Hexagon



Circle

6. Are you symmetrical? Is a ship symmetrical? An automobile? An animal? Where is the axis of symmetry in a front-view photograph of a person?

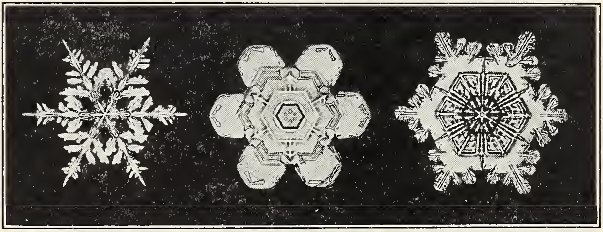


Courtesy American Museum of Natural History.

BUTTERFLIES

Most animals are symmetrical, but the butterfly is a particularly striking example.

7. The butterfly is a beautiful example of a symmetrical creature. Can you find the axis of symmetry?



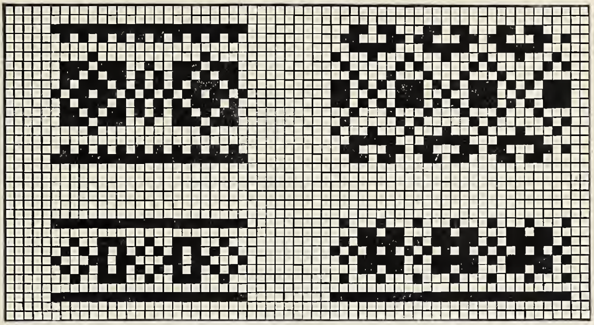
Photographs by W. A. Bently.

SNOWFLAKES

Plato said: "God continually geometrizes."

8. How many axes of symmetry can you find in these snowflakes? Are they beautiful?

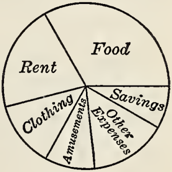
9. Are the needlework figures on the following page symmetrical? How many axes of symmetry can you find in them?



SYMMETRY IN NEEDLEWORK
DESIGN

THE CIRCLE GRAPH

The circle graph is used to show how the whole of anything is distributed into its parts. It should be used only when all of the data occur at the same time and when the sum of all the data has a definite meaning.



This graph shows how an average family uses its income. Here the sum of all the parts is the total income.

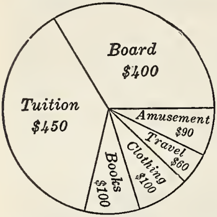
About what part of the income is spent for food? For clothing? For amusements?

If the income of this family is $1800, estimate the amount spent for amusement. How much is saved?

What is the largest expense? The smallest?

How to make a circle graph.

Illustration: A college student spends $400 for board, $450 for tuition, $100 for books, $100 for clothing, $60 for travel, and $90 for amusements.



Solution. First find the total amount spent. Then find the part of 360° that each number is of this total. 400 is represented by $\frac{400}{1200} \times 360° = 120°$. $\frac{450}{1200} \times 360° = 135°$, and so on.

400	120°
450	135°
100	30°
100	30°
60	18°
90	27°
1200	360°

Now lay off at the center of a circle angles of 120°, 135°, 30°, etc. Finally label the parts of the circle, and shade or color them.

Notice that the number of degrees N in the arc can be found by using the formula $N = \frac{360\,I}{T}$, where I is the value of an item and T is the total of all items.

Find N when:

(*a*) $I = 240$ and $T = 480$ (*c*) $I = 16$ and $T = 384$
(*b*) $I = 82$ and $T = 246$ (*d*) $I = 86$ and $T = 720$

Exercises

Make a circle graph for each of these:

1. On school days George is in school 6 hrs., sleeps 8 hrs., eats 2 hrs., studies 3 hrs., works 2 hrs., and plays 3 hrs.

2. In an algebra test 7 pupils received a mark of A; 8, a mark of B; 10, a mark of C; 5, a mark of D; and 6 failed.

3. In our high school there are 200 seniors, 250 juniors, 280 sophomores, and 350 freshmen.

4. A cost accountant found that the expenses in a manufacturing business, based on each dollar spent, were distributed as follows: wages, 45¢; cost of materials, 30¢; cost of sales, 12¢; overhead, 10¢; and profit, 3¢.

5. In making his income-tax report, Mr. Wells found that he had $2000 salary, $800 interest on investments, and $200 other income.

6. A family budget provides 30% for food, 25% for rent, 15% for clothing, 20% for other expenses, and the remainder for savings.

7. In a single year the U.S. spent $960,000,000 for veterans' relief, $700,000,000 for defense, $400,000,000 to reduce its debt, $600,000,000 interest, $440,000,000 for farm relief, and $1,100,000,000 for all other expenses.

8. The land area of the earth is divided as follows: Asia, 17,000,000 sq. mi.; Africa, 11,000,000 sq. mi.; No. America, 8,000,000 sq. mi.; So. America, 7,000,000 sq. mi.; Europe, 3,500,000 sq. mi.; and Australia and islands, 3,500,000 sq. mi.

9. The grain crops of the world, in 100,000,000 bu., for a certain year were: corn, 44; wheat, 50; oats, 45; barley, 20; and rye, 21.

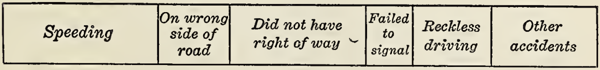
10. Here is the way a certain city spent its income one year: education, $78,000; protection, $92,000; interest on bonds, $120,000; other expenses, $70,000.

Which is the largest expense? If this expense were not necessary, by about what fraction would the taxpayers' burden be decreased?

11. One year our exports to the Latin American countries amounted to $900,000,000. $180,000,000 of it went to Argentina, $100,000,000 to Brazil, $50,000,000 to Chile, $170,000,000 to Cuba, $120,000,000 to Mexico, and the remainder to the other countries.

THE RECTANGLE DISTRIBUTION GRAPH

Instead of using the circle, we sometimes divide a long rectangle into parts to show how a whole is distributed. This graph shows the distribution of automobile accidents.



What do you think is the cause of the largest number of accidents? About what per cent of the accidents does it cause? What is the next most important cause of accidents? What per cent of accidents does this cause?

Exercises

Make a rectangle distribution graph for these data:

1. The number of million people in the world is: No. America, 150; So. America, 65; Europe, 475; Asia, 1000; Africa, 140; others, 10.

2. The percentage population of the United States according to age is: under 5 yrs., 10%; 5 to 14 yrs., 20%; 15 to 20 yrs., 10%; 21 to 45 yrs., 38%; and over 45 yrs., 22%.

3. Automobile accidents to pedestrians (people walking) are divided as follows: crossing against signal, 11%; crossing where there is no signal, 15%; crossing not at the corner, 25%; children playing in the street, 18%; coming from behind parked car, 13%; and all other causes, 18%.

If accidents happened to about 300,000 pedestrians, about how many children were hit while playing in streets? How many were hit coming from behind parked cars?

4. Of the world's oil supply, the United States produces 63%; Europe, 15%; So. America, 12%; Asia, 5%; and the rest of the world, 5%.

5. In 1930 the number of million people in the United States whose parents were native was 82; whose parents were foreign, 17.5; one of whose parents was foreign, 8.5; who were themselves born in a foreign country, 14.

6. The cost of building a house was distributed as follows: foundation, 10%; lumber, 25%; carpenter work, 25%; mason and plumbing, 20%; and finishing, 20%.

7. In a recent year the number of million children in school in the United States was: in public elementary, 21; in public high, 5; in all other schools, 6.

Review Exercises

1. Explain how a line segment differs from a line.

2. How many points are needed to determine a straight line?

3. What is the difference between constructing and drawing a figure?

4. How many degrees are there in a right angle? In a straight angle?

5. What is the name of the kind of angle that has 90°? 70°? 130°? 180°?

6. Are all acute angles equal? Are all right angles equal? Are all obtuse angles equal?

7. Explain the difference between perpendicular and vertical.

8. Write in short form: $a + a + a + a + a$; $a \cdot a \cdot a \cdot a \cdot a$; $aaaaaaa$.

9. Draw a line segment, and construct one twice as long.

10. Draw an angle, and construct one 3 times as large.

11. At 5 o'clock the hands of a watch form an angle of how many degrees?

12. Find the supplement of an angle of 105°, of 75°.

13. How can you tell if an object is symmetrical?

14. Is a tree symmetrical? Do you consider a nearly symmetrical tree more beautiful than one that lacks symmetry?

15. In the formula $K = abC$, find K when $a = 12$, $b = 20$, and $C = .88$.

16. In the formula $S = a + ar + ar^2$, find S when $a = 5$ and $r = 3$.

17. Make a rectangle graph to show the following: The sales in the departments of a store were: groceries, $60,000; clothing, $50,000; hardware, $18,000; toys, $32,000; and furniture $40,000.

18. In the formula $m = \frac{c(1 + p)}{1 - d}$, find m when $c = \$8.80$, $p = 40\%$, and $d = 20\%$.

Honor Work

Find the value of the letter on the left side in these exercises:

19. $x = \frac{P_1x_1 + P_2x_2}{P_1 + P_2}$ $P_1 = 120$, $P_2 = 80$, $x_1 = 9$, and $x_2 = 11$

20. $x = \frac{3h_1^2 + 2h_1h_2 + h_2^2}{2(2h_1 + h_2)}$ $h_1 = 10$ and $h_2 = 15$

21. $P = p + (h + x)y$ $p = 42.5$, $h = 6.38$, $x = 5.12$, $y = 20$

22. $P = \frac{r(p - p_1)}{t}$ $r = 8.76$, $p = 13.4$, $p_1 = 2.4$, $t = .05$

23. $M = \frac{Ip}{e}$ $p = 3460$, $I = 6.38 \times 10^8$, $e = 2 \times 10^4$

Computation Test

1. $h = d + e$ $d = 43$ and $e = 5$ $h =$

2. $A = \frac{1}{2} bh$ $b = 24$ and $h = 15$ $A =$

3. $V = \frac{1}{3} \pi r^2 h$ $\pi = 3\frac{1}{7}$, $r = 14$, $h = 12$ $V =$

4. $F_1 = \dfrac{F_2 v_1}{v_2}$ $F_2 = 60$, $v_1 = 55$, $v_2 = 33$ $F_1 =$

5. $P = \dfrac{pT^3}{t^3}$ $p = 18$, $T = 4$, and $t = 2$ $P =$

6. $V = V_0 \dfrac{(273 + t)}{273}$ $V_0 = 70$ and $t = 39$ $V =$

7. $V = \frac{1}{6} h(b^2 + t^2 + 4\,M)$ $h = 20$, $b = 7$, $t = 5$, $M = 13$ $V =$

8. $n = .16\left[\dfrac{2\,bc}{.36\,h(B + b)} - 1\right]$ $c = 8$, $b = 10$, $B = 30$, $h = 16$ $n =$

Matching Test

Match the number of the statement with the letter before the word that completes the sentence:

1. The supplement of an angle of 150°
2. An angle greater than a right angle
3. A straight line is determined by . . .
4. The point of an angle is called its . . .
5. When lines meet at right angles, they are . . .
6. A kind of graph used for distributing the parts of a whole
7. A shorthand way of writing a rule
8. Distance around a figure
9. A small number that tells how many times a quantity is taken as a factor
10. An angle less than a right angle

a. Perpendicular
b. Formula
c. Circle
d. 30°
e. Vertex
f. Acute angle
g. Exponent
h. 2 points
i. Perimeter
j. Obtuse angle

Construction Test

1. Draw an angle of about 30°, and construct an angle 4 times as large.

2. By starting with a straight angle, construct an angle of 45°.

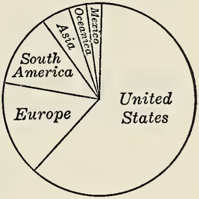
3. Construct a line perpendicular to another line at a point A of that line.

4. Construct an angle $1\frac{1}{2}$ times an angle that you start with.

5. Draw a line segment, and construct a segment $2\frac{1}{2}$ times as long.

Test on Circle Graphs

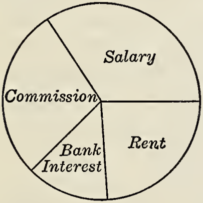
1. This graph shows the distribution of oil among the countries of the world.



(*a*) The United States produces about . . . of the oil.

(*b*) The continent that produces the second largest amount of oil is . . . .

2. This graph shows the distribution of a man's income. His total income from salary, commission, interest, and rent was $6000.



(*a*) His salary was about . . . .

(*b*) He received about equal amounts from . . . and . . . .

(*c*) He received about . . . interest.

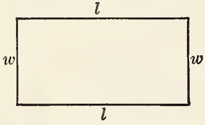
(*d*) The fraction of his income from rent was about . . . .

3. Make a circle graph showing these statistics: State taxes are distributed as follows: on property, 22%; on income and inheritance, 11%; on automobiles and gas, 27%; on other licenses, 14%; miscellaneous, 26%.

Chapter 4

ALGEBRA AS A LANGUAGE

What we can do with letters. To find the perimeter p of the rectangle we use the formula $p = l + w + l + w$ where l means the length and w the width. Now $w + w = 2\,w$ for w alone always means $1\,w$. In algebra just as in everyday speech, we often omit the number 1. We say "three books" when we mean 3 books, but we generally say "the book" when we mean 1 book. The "one" is omitted. Also $l + l = 2\,l$. So our formula becomes $p = 2\,l + 2\,w$. We cannot add the l and w until we know what numbers they stand for, so we leave it in the form $2\,l + 2\,w$.



To multiply w by 3, we write $3\,w$, for we agreed that when we wrote two quantities together without a sign between them, we meant to multiply them. But when we wish to add 3 to w, we must leave it in the form $w + 3$ until we know the value of w.

Quantities that are connected by multiplication only are called *factors*. 3 and w are factors of $3\,w$. Each factor of a product is the *coefficient* of the others, but we more often use the word coefficient to refer to the number only. Therefore, 3 is the coefficient of w in $3\,w$.

An expression made up of numbers and letters, connected by multiplication and division only, is called a term. Thus, $3\,w$ is a term, but $w + 3$ is not a term because the parts are connected by addition. When you come to a plus or minus sign, you have come to the end of a term. There are then two terms in $w + 3$. When two terms are composed of the same powers of the same letters, we call them *like terms.* Like terms may have different numbers as coefficients. For example, $5\,ab$ and $3\,ab$ or a^2b^3 and $5\,a^2b^3$ are like terms.

We have also seen that when we have two or more factors that are equal, we can write the factor once and put a small number to the right and above to tell how many of those factors there are. For example, $aa = a^2$, $aaa = a^3$, $aaaa = a^4$, and so on. A small number written to the right and above a quantity to tell how many times it occurs as a factor is called an ***exponent.***

Be careful not to confuse terms with factors, and coefficients with exponents.

If $w = 3$, then:
Terms: $w + 5 = 3 + 5 = 8$
Factors: $5\,w = 5 \times 3 = 15$
$w + 5$ is not the same as $5\,w$
If $k = 4$, then:
Coefficient: $3\,k = 3 \times 4 = 12$
Exponent: $k^3 = 4 \times 4 \times 4 = 64$
$3\,k$ is not the same as k^3

Exercises

1. Give the terms in each of these expressions, and also give the factors in each term:

(*a*) $4\,a$
(*b*) $6\,a^2b$
(*c*) $d + 1$
(*d*) $2 + 5\,cx^3$
(*e*) $3\,abcde - 1 + 2\,f$
(*f*) $4\,x^4y^2z^3 - 2$

2. State which numbers are coefficients and which are exponents in Exercise 1.

3. If $a = 10$ and $b = 3$, find the value of:

(*a*) $2\,b$
(*b*) $a + 4$
(*c*) b^2
(*d*) $a^2 + 71$
(*e*) $4\,b^3$
(*f*) $a + b - 4$
(*g*) ab
(*h*) $7\,a^2 - 3\,ab + 2$
(*i*) $5\,a^4 + 2\,ab$

4. If $a = 1$, how much is $3\,a$? a^3?

5. If $b = 2$, how much is $3\,b$? b^4? $5\,b^3$? b^6?

6. If $c = 0$, how much is $3\,c$? $c + 5$? c^3?

THE LANGUAGE OF ALGEBRA

Algebra is a language that can say more in a small space than any other language in the world. Compare it with English.

English. To find the amount at simple interest, we multiply together the principal, the rate, and the time, and add the result to the principal.

Algebra. $A = P + Prt$

Illustration: Write a number that is 5 less than the sum of x and y.

Answer. $x + y - 5$

Exercises

1. What is the sum of a and 3? Of x and y?

2. What is the product of y and 4? Of a and b?

3. Express 7 more than x, a more than 5, c more than d.

4. How many inches are there in 2 ft.? In f ft.? In 5 yds.? In y yds.?

5. How many inches are there in 2 ft. and 7 in.? In f ft. and 7 in.? In b ft. and c in.?

6. Dorothy had c¢ and spent 15¢. How many cents has she left?

7. What number is 7 less than n? x less than 12? b less than a?

8. Express in cents: 3 dimes, d dimes, 2 quarters, q quarters, 2 quarters and 3 dimes, q quarters and d dimes.

9. The difference of two numbers is 5, and the smaller is 8. What is the larger? What did you do to find the larger?

10. The difference of two numbers is d, and the smaller is s. What is the larger?

11. Write in algebra: a increased by b, x diminished by y, 3 times the sum of a and b, twice the product of m and n.

12. John is 3 yrs. older than Robert. Express John's age if Robert is r yrs. old. Express Robert's age if John is j yrs. old.

13. How many pounds are there in 48 oz.? In k oz.?

14. How many feet are there in 24 in.? In i in.?

15. What is the quotient of 3 divided by 8? Of n divided by 8? Of a divided by b?

16. Find the cost of p pencils at 5¢ each, of b books at c¢ each, of e eggs at 50¢ a doz.

17. Gladys saves 50¢ a week. How much will she save in 14 weeks? In w weeks? In d days?

18. How far will an automobile travel in 5 hrs. at 35 mi. an hr.? In h hrs. at 35 mi. an hr.? In 5 hrs. at m mi. an hr.? In h hrs. at m mi. an hr.?

19. How many seats are there in a classroom that has 5 rows of 7 seats each? 5 rows of s seats each? r rows of 7 seats each? r rows of s seats each?

20. By how much does 17 exceed 12? What did you do to 17 and 12 to obtain the answer?

21. By how much does 17 exceed x? k exceed 12? m exceed n?

22. When you have already expressed two numbers, how do you express the amount by which one exceeds the other? Express the amount by which a exceeds b.

23. What number exceeds 8 by 7? What did you do to 8 and 7 to obtain the answer?

24. What number exceeds 8 by x? a by 7? m by n?

25. When you know one of two numbers and the amount by which the other exceeds it, how do you express the other? Write a number that exceeds h by k.

26. What number is 5 less than 9? Do you subtract 9 from 5 or 5 from 9?

27. What number is x less than 9? 5 less than y? x less than y? When you express "is less than" in algebra, do you write the algebraic symbols in the same order in which the words occur in English?

28. How long will a train take to travel 120 mi. at 30 mi. an hr.? m miles at r mi. an hr.?

Translation Exercises

Translate from algebra into English:

1. $h = 2a$, if h is the number of dolls Helen has and a the number Alice has

2. $r = w + 5$, if r is the number of marbles Roy has and w is the number William has

3. $g + 7 = 3b$, when g is George's age and b is Ben's, in years

4. $v = lwh$, when v is the volume of a box, l its length, w its width, and h its height

5. $i = prt$, if i is the interest on p dollars at rate r for t yrs.

6. $C = A - L$, when A stands for assets, L for liabilities, and C for capital

7. $d = rt$, when r is the rate at which a train is traveling, t the time it is traveling, and d the distance it goes

8. $p = 3\,e - 10$, when p is the number of cents Paul has and e the number Emily has

9. $F = \frac{9}{5}\,C + 32$, when F is the number of degrees registered on a Fahrenheit thermometer and C the number on a centigrade thermometer

10. $c = 15 + 20\,m$, when c is the cost of riding m miles in a taxi

Translate from English into algebra, using the initial letters of the important words as symbols:

11. The gain (g) is found by subtracting the cost (c) from the selling price (s).

12. The area of a rectangle equals the product of its length and width.

13. Three times a certain number (n) is 6 more than twice the number.

14. If 13 is added to 4 times a number, the result is 31 more than the number.

15. The volume (v) of a pyramid is $\frac{1}{3}$ the product of its base (b) and height (h).

16. The cost (c) of sending a package by parcel post for a certain zone is 6¢ more than the number of pounds (n).

17. The circumference of a circle equals two π times the radius.

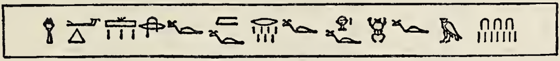
18. If a stone is dropped from a height, the distance (d) it will fall is 16 times the square of the number of seconds (t).

19. The average (a) of three numbers—x, y, and z—is found by adding them and then dividing their sum by 3.

20. The length (l) of the belt that will go around 2 equal pulleys is $2\,\pi$ times the radius (r) of a pulley added to twice their distance apart (d).

THE EQUATION

Away back in the distant past, 1700 years before Christ, an Egyptian priest named Ahmes solved problems by



algebra, but he did not know how to use the beautiful shorthand forms that you are learning now. For example, the equation shown here is simply

$$\frac{2x}{3} + \frac{x}{2} + \frac{x}{7} + x = 37$$

Can you find the denominators 3 and 7 and the number 37?

An ***equation*** is a statement that two expressions are equal.

Formulas, such as $A = lw$, $V = \pi r^2 h$ and $A = P + Prt$, are equations. But there are also other kinds of equations, such as $x + 3 = 7$, $2\,y - 5 = 4$, $3\,x - 2\,y = 9$. Equations are very important. Much of our work in algebra will be studying equations and learning how we can use them to help us solve problems. Finding the answer to an equation is called ***solving the equation,*** and the answer is the ***root*** of the equation.

You may think of an equation as a question. For example, $x + 5 = 9$ may be read, "What number added to 5 gives 9?" Evidently the answer is 4, so $x = 4$.

Check the answer by putting 4 in place of x in the original equation.

$$4 + 5 \mid 9$$
$$9 = 9$$

Solve $\frac{1}{4}\,x = 3$. The question is: One-fourth of what number is 3? Then x must be 4×3 or 12.

Many easy equations can be solved by arithmetic. We shall, however, learn methods of solving equations that are too difficult to be done this way.

Class Exercises

Solve the following equations by arithmetic, and check your answers:

1. $x + 4 = 7$
2. $x + 8 = 9$
3. $x - 1 = 2$
4. $x - 3 = 10$
5. $3x = 12$
6. $7x = 21$
7. $x - 3 = 7$
8. $x + 2 = 9$
9. $x - 7 = 2$
10. $x - 2 = 6$
11. $5x = 25$
12. $x + 5 = 6$
13. $\frac{x}{2} = 5$
14. $\frac{x}{4} = 3$

Optional Exercises

15. $2x + 3 = 11$
16. $3x - 4 = 8$
17. $\frac{x}{2} - 5 = 3$
18. $5x - 9 = 6$
19. $7x + 3 = 17$
20. $4y - 1 = 11$
21. $\frac{x}{5} + 2 = 3$
22. $6z - 8 = 5z$

Honor Work

23. $3x - 5 = x + 6$
24. $7x - 5 = 2x + 10$
25. $3 + x = \frac{x}{3} + 5$
26. $x + a = b$
27. $cx - d = 2$
28. $\frac{x}{4} = 2a$

Practice in Algebra Shorthand

How fast can you write shorthand? Try to write the equation for each of these sentences while it is being dictated by your teacher.

1. What number increased by 11 equals 15?

2. Three times a number diminished by 9 equals 6.

3. Twice a number exceeds 8 by 5.

4. If twice a number is increased by 4, the result is 3 times the number.

5. Twelve exceeds a certain number by 7.

6. If 8 is added to 5 times a number, the result is 13.

7. If 3 times a number is diminished by 11, the remainder is 10.

8. Five times a number equals 12 diminished by the number.

9. If a certain number is divided by 3, the quotient is 7.

10. If 4 times a number is increased by 5, the result is 7 less than 6 times the number.

11. If I double a number and add 10, the result is 18 more than the number.

12. Three-fourths of a number increased by 4 equals 16.

13. If 7 is added to 3 times a number, the result is 1 more than 5 times the number.

14. Three times a number exceeds 10 as much as the number exceeds 2.

15. If 5 is added to 7 times a number, the result is 13 diminished by the number.

16. A number equals $\frac{1}{3}$ of itself increased by 8.

17. I am thinking of a number. If I double it and subtract 4, I shall have 3 more than the number.

18. Fifteen exceeds a number as much as the number exceeds 3.

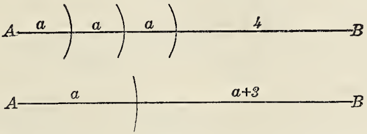
19. Seven times a number lacks 12 of equalling 9 times the number.

20. If 17 is added to 4 times a number, the result is 3 more than 6 times the number.

Algebra Shorthand in Geometry

1. The line AB is 10 in. long. After a length a has been cut off 3 times, the part left is 4 in. long.

2. A line AB, 17 in. long, is cut into 2 parts. One part is a and the other part is 3 in. longer than a.



3. $\angle ABC$ equals 65°. After 2 equal angles, each x, are cut off, the part left has 25°.

4. After 2 equal angles, each y, were laid off, it required 40° more to make a straight angle.

5. I laid off 5 equal angles, and their sum was a straight angle.

Review Exercises

1. Translate into algebra:

(*a*) Twice a number decreased by 13 leaves 7.

(*b*) If a number is added to 3 times itself, the result is 36.

(*c*) Divide a certain number by 3, and you have 10 less than the number.

2. Express in algebra:

(*a*) A number 3 larger than n

(*b*) A number 4 times as large as t

(*c*) Katherine's age 3 yrs. ago, if k is her age now

3. An angle has $3x$ degrees, and its vertical angle has $2x + 50$ degrees. Find x, and also the number of degrees in the angle.

4. The change in pressure in a liquid is given by the formula $P_2 - P_1 = dg(h_2 - h_1)$. Find P_2 when $P_1 = 17$, $d = .9$, $g = 32$, $h_2 = 20$, and $h_1 = 4$.

5. Draw an angle of about 20°, and construct another angle 5 times as large.

6. Construct a line perpendicular to another line at a point on the line.

7. Make a formula for:

(*a*) The kinetic energy (k) of a moving body is found by multiplying $\frac{1}{2}$ its weight (w) by the square of its velocity (v).

(*b*) The amount of work (w) done in raising a weight is the product of the force (f) and the distance (d) through which it moves.

8. Find g in the formula $g = \dfrac{4\pi^2 l}{t^2}$, if $l = 7$ and $t = 3$.

9. Make a bar graph to illustrate these figures: The specific heat of water is 1; of alcohol, .7; of benzine, .9; of ether, .5; and of sulphuric acid, .3.

10. Make a graph for the formula $w = 100d$ from $d = 0$ to $d = 8$.

11. The girls of a club saved in a single month these amounts: \$2.00, \$.50, \$2.50, \$3.00, \$1.50, \$3.00, \$.50, \$2.50, \$1.50, \$2.50, \$1.00, \$6.00, \$2.00. What was the average, the median, and the mode?

12. The cost of sending a telegram is $c = 25 + 3(n - 10)$. What is the cost of sending a 35-word message?

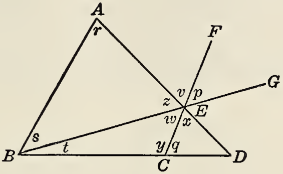
Completion Test

1. The perimeter of a square whose side is s is . . . .
2. The number of cents in d dimes and n nickels is . . . .
3. In 3 yrs. a girl y yrs. old will be . . . yrs. old.
4. The number of days in w weeks is . . . .
5. The cost of p pencils at c¢ each is . . . .
6. The area of a cube, whose edge is e, is . . . .
7. The next whole number bigger than n is . . .
8. An angle contains $a + 40$ degrees. Its supplement contains . . . degrees.
9. If pencils cost 5¢ each, the number of pencils I can buy for c¢ is . . . .
10. Ted traveled m mi. in 5 hrs. At that rate in 7 hrs. he would travel . . . mi.

Test on Angles

The small letter representing angle:

1. BAE is . . . .
2. ABE is . . . .
3. BEC is . . . .
4. EBD is . . . .
5. FCB is . . . .
6. GBD is . . . .



7. The vertical angle to p is . . . .
8. The supplement of q is . . . .
9. The supplement of DEB is . . . .
10. The supplement of $v + p$ is . . . .

Construction Test

1. Construct an angle of 45°.
2. Construct a line 3 times a given line.
3. Cut a line into 4 equal parts.
4. Construct an angle twice a given angle.

Formula Test

Make formulas for these rules:

1. The selling price equals the cost plus the gain. **1.** . . .

2. The perimeter of a square is 4 times a side. **2.** . . .

3. In making coffee the number of tablespoonfuls (t) to use is 1 more than the number of people (p). **3.** . . .

4. To find the profit, subtract the cost and the expenses from the selling price. **4.** . . .

5. The distance an airplane flies is the product of the rate and the time. **5.** . . .

6. The average of three numbers, x, y, and z, is their sum divided by 3. **6.** . . .

Test in Solving Equations

Find the value of the letter in each equation:

1. $x + 2 = 5$
2. $x + 8 = 33$
3. $x - 5 = 9$
4. $x - 3 = 16$
5. $2x = 6$
6. $5x = 30$
7. $\frac{x}{2} = 7$
8. $\frac{x}{3} = 5$
9. $2x + 3 = 11$
10. $3x + 7 = 22$
11. $3x - 5 = 7$
12. $5x - 3 = 12$
13. $\frac{x}{3} + 2 = 6$
14. $\frac{x}{4} + 3 = 10$
15. $1.4x = 14$
16. $3.6x = 7.2$
17. $3x = 10 + x$
18. $7x = 15 + 2x$
19. $\frac{x}{2} + \frac{x}{3} = 10$
20. $\frac{x}{2} - \frac{x}{4} = 5$

Chapter 5

ALGEBRAIC ADDITION

ALGEBRAIC NUMBERS

THE thermometer in ordinary use in the United States is the Fahrenheit, on which the freezing point is 32° and the boiling point 212°. But in most of Europe and in scientific work in this country, a different thermometer, the centigrade, is used. This has the freezing point at 0° and the boiling point at 100°. One day while I was traveling in France, I noticed that the temperature was 20° centigrade. Do you think it was a hot day or a cold day? You can find out by using the formula $F = \frac{9}{5}C + 32$ where F is the reading on our Fahrenheit thermometer and C that on the centigrade thermometer.

What Fahrenheit reading corresponds to 40° centigrade? To 10° centigrade? To 80° centigrade?

And when we wish to find the centigrade reading corresponding to one on the Fahrenheit, our formula can be changed to the form:

$$C = \tfrac{5}{9}(F - 32)$$

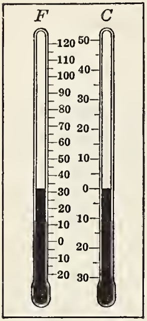
Copy the following table on a separate sheet, and complete it, using this formula:

F	95°	86°	77°	59°	41°	32°	23°	14°
C								

$$\begin{aligned}\text{When } F = 95,\ C &= \tfrac{5}{9}(95 - 32)\\ &= \tfrac{5}{9} \times 63\\ &= 35\end{aligned}$$

Now you fill in the others.

When we come to 23°, we have $C = \frac{5}{9}(23 - 32)$. What does this mean? Can we take 32 from 23, or is there no centigrade reading corresponding to 23° Fahrenheit? Look at the thermometers. When the Fahrenheit reading was 32°, what was the centigrade? If then the temperature drops below 32°, where would you expect to find the centigrade reading? On our thermometer, 23° F. corresponds to 5° below zero centigrade. Then in some way our formula should express "5 below zero." We shall now learn how to deal with this new kind of number.



A new kind of number. Look at either thermometer. Does the scale end at 0, or are there numbers on the other side of 0? Can you think of other places where the numbers can go beyond 0? Later you will find many uses for a number scale that does not end at 0.

In algebra there are numbers on both sides of 0. On one side are the ***positive numbers.*** These are the ordinary numbers that you have studied in arithmetic. They may be preceded by a plus sign or they may be written without any sign at all. Numbers on the opposite side of 0 from positive numbers are called ***negative numbers.*** They are always preceded by a minus sign. Positive and negative numbers are used to indicate opposites. Such numbers are called ***algebraic numbers.***

What is the opposite of:

1. A gain of $20?
2. 40° north latitude?
3. A 15° rise in temperature?
4. 75° west longitude?
5. 3000 ft. above sea level?
6. Going forward 10 yds?
7. Earning $1.75?
8. $800 profit?
9. Adding 7?
10. 422 A.D.?
11. Going 40 mi. east?
12. Gaining 10 lbs. weight?

13. In music, what is the opposite of a sharp? What is meant by − 3 sharps?

Class Exercises

What is meant by:

1. A temperature of − 40°? Of + 17°?

2. A latitude of − 15° if north latitude is positive?

3. The date − 44 if time after Christ is positive?

4. A gain of − $100?

5. A longitude of − 60° if west longitude is positive?

6. A gain in weight of − 2 lbs.?

7. A net change of − $2\frac{1}{2}$ in the price of a stock?

8. A deposit of − $28 in a savings bank?

9. A gain of − 15 yds. by a football team?

10. A score of − 220 points? Of + 150 points?

11. A man's property is + $500? Is − $800?

12. The temperature changed − 12°? + 10°?

13. The altitude above sea level of Death Valley, California, is − 276 ft.?

What is the final result of:

14. Winning 200 points in a game and afterwards losing 250 points?

15. Walking 2 mi. north and then 5 mi. south?

16. Earning $2.30 one day and then spending $3.40 the next day?

17. A rise of temperature of 8° from − 12°?

18. A ship starting at 25° south latitude and traveling 40° north?

Optional Exercises

19. Will a balloon weighing − 600 lbs. rise if the weight carried is + 400 lbs.? Is + 700 lbs.?

20. A firm has assets of $7000. What is it worth if its liabilities are $5000? If $7000? If $10,000?

21. At what rate can Paul row up-stream if he can row 4 mi. an hr. in still water and the current is flowing 3 mi. an hr.? If the current is flowing 6 mi. an hr.?

22. Dorothy has 40¢ and owes 65¢. What is her balance?

23. Augustus Caesar ruled Rome from the year -31 until the year $+14$. What does this mean? How many years did he rule?

24. At noon the temperature was 24°. During the afternoon and night it dropped 34°, but rose 10° again the next morning. What was the temperature the next noon?

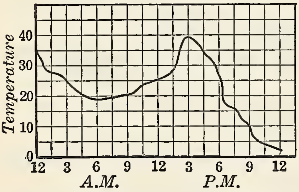
25. Gladys is given \$4, then spends \$10, and later earns \$9. How much money has she?

26. Here is a section of a stock-exchange report for a certain year as taken from a daily newspaper. The net-change column shows how much higher the stock was at the time reported than at the close of the previous year. Explain each net change, and tell what the price was a year earlier:

	Last	*Net Change*
Allis Chalmers	$190\frac{1}{2}$	$+73\frac{1}{2}$
American Can	$110\frac{3}{8}$	$+35\frac{3}{8}$
American Car and Foundry	$98\frac{1}{8}$	$-12\frac{1}{8}$
American Locomotive	$108\frac{3}{4}$	$-3\frac{5}{8}$
American Telephone and Telegraph	193	$+14\frac{3}{8}$

27. A recording thermometer made the following graph on a winter day:

(*a*) What was the temperature at 3 A.M.? At 6 A.M.? At 12 noon? At 9 P.M.?



(*b*) At what hour was it coldest? Warmest?

(*c*) Tell at what times the temperature was 30°, 15°, 10°.

(*d*) What was the change in temperature from 3 A.M. to 6 A.M.? From noon to 3 P.M.? From 6 P.M. to 9 P.M.? From 9 P.M. to midnight?

(*e*) When was the temperature increasing most rapidly? Decreasing most rapidly?

28. When in the stratosphere 14 mi. above the earth, Stevens and Anderson reported the temperatures as "55° below zero centigrade outside and $-7°$ centigrade inside." If $F = \frac{9}{5}C + 32$, what were those temperatures on a Fahrenheit thermometer?

29. Is the temperature increasing or decreasing when it changes from: (*a*) $+10°$ to $+40°$? (*b*) $+20°$ to $-30°$? (*c*) $-18°$ to $+8°$? (*d*) $-20°$ to $-5°$?

30. Can a man's wealth increase to 0?

Honor Work

Deviations from the average. In statistics it is often important to know not only the average of several items, but also how much each item differs from the average. When an item is greater than the average, this deviation is positive; when less than the average, the deviation is negative.

In these exercises, find the average, then the deviation of each item from the average, and finally the sum of the deviations:

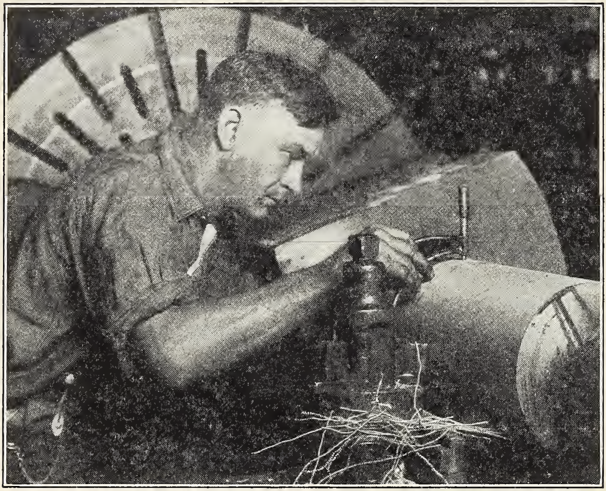
31.	**32.**	**33.**	**34.**
8	32	18	82
12	43	19	67
9	27	21	91
7	51	23	73
4	36	16	85
11	44	11	96
5	33	18	87

35. For lunch one week Sarah spent 32¢, 26¢, 20¢, 30¢, and 22¢. What was her average, and how much did she deviate from the average on the various days?

36. Mr. Wright's weekly pay for 2 mo. was \$43, \$52, \$40, \$44, \$47, \$53, \$38, and \$35. Find his average weekly pay, the deviations from the average, and the sum of the deviations.

37. A scientist wished to know a distance very accurately so he measured it 10 times. If his measurements were 42.3 cm., 42.1 cm., 42.4 cm., 42.0 cm., 42.4 cm., 42.2 cm., 41.9 cm., 42.2 cm., 42.3 cm., and 42.2 cm., find the average and the deviations from the average. What is the sum of the deviations?

The graph scale extended. When we studied graphs, we began our scale at 0 and laid off the numbers to the right and upward in order from that point. In dealing with temperatures we used only those that were above 0. But temperatures can be below 0, and it is also convenient to be able to extend our scale to the left of the 0 on the horizontal axis.



Photograph by Lewis W. Hine.

MACHINIST USING MICROMETER

The machinist, as well as the scientist, must measure very accurately, sometimes to the ten-thousandth of an inch.

−8 −7 −6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6 7 8

These numbers represent points on a straight line that can run on indefinitely in both directions.

If you start at point 2 and add + 3, what point do you reach? Do you go to the right or to the left when you add a positive number?

If you begin at point $+6$ and add -2, what point do you reach? Do you go to the right or to the left when you add a negative number?

Add $+4$ to -5. Add -5 to $+4$. Did you end at the same point both times?

Exercises

By starting at the first number named and going in the proper direction, perform these additions graphically:

1. $(+2)+(+3)$
2. $(+5)+(+1)$
3. $(+6)+(-1)$
4. $(+2)+(-5)$
5. $(-3)+(+4)$
6. $(-7)+(+3)$
7. $(+5)+(-2)$
8. $(-5)+(-1)$
9. $(+0)+(-2)$
10. $(0)+(-6)$
11. $(+2)+(-7)$
12. $(+0)+(-0)$

Determine from the graph what you must add to the first number to get the second number:

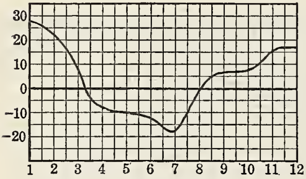
13. $+1$, $+5$
14. -2, $+4$
15. 0, $+6$
16. -3, $+4$
17. -2, -6
18. 0, -3
19. $+5$, -1
20. -6, $+3$
21. -2, -2

THE ALGEBRAIC NUMBER IN SURVEYING

When a surveyor wishes to make a contour map showing elevations, he first selects a starting point or bench mark, which he marks 0 elevation. Whenever it is convenient, sea level is taken as 0, but any other starting level will do. Then he measures the elevations at equal horizontal distances, say 10 ft., calling the points "Station 1," "Station 2," etc. If a point is 5 ft. higher than his starting point, he labels it $+5$; if 3 ft. lower, -3. He records these readings in his note-book as follows:

Station	*Elevation*	*Station*	*Elevation*	*Station*	*Elevation*
1	+ 28	5	− 10	9	+ 6
2	+ 22	6	− 12	10	+ 7
3	+ 7	7	− 18	11	+ 15
4	− 8	8	0	12	+ 16

Back at the office, this graph is made from his readings. In the graph, 0 is the level of the river.



1. Which bank of the river is higher?

2. At what station is the deepest channel of the river?

3. Between which stations is the river bed nearly horizontal?

4. Which bank is steeper above water?

5. If the stations are 10 ft. apart, how wide is the river?

6. Would it be safe for children who cannot swim to bathe in this stream?

Exercises

Make a contour map for these elevations:

	Station	*Elevation*	*Station*	*Elevation*	*Station*	*Elevation*
1.	1	0	5	− 4	9	+ 29
	2	− 5	6	0	10	+ 20
	3	− 1	7	+ 10	11	+ 31
	4	+ 3	8	+ 18	12	+ 35

	Station	*Elevation*	*Station*	*Elevation*
2.	1	+ 100	6	+ 360
	2	+ 170	7	+ 390
	3	+ 230	8	+ 90
	4	+ 380	9	− 10
	5	+ 490	10	− 100

HISTORICAL NOTE ON NEGATIVE NUMBERS: Although the laws of negative numbers were known from early times, these numbers were not really understood until about 1500. Diophantus, about 275 A.D., worked with them, but called the answers absurd. Men like Vieta, Fermat, Harriot, Stifel and Hudde helped clear up the idea, but it was the geometric representation of Descartes that finally gave them a definite meaning.

© *G. A. Douglas from Gendreau, N.Y.*

GIRL USING AN ADDING MACHINE

Most business houses today use machines like this to save time and effort.

Review of arithmetic. *Add:*

1. 6289
4251
1273
7654
3059
6428
5930
4217

2. 5166
3842
7903
6021
5199
4874
3333
5814

3. 1998
3677
5914
8719
2324
3658
7219
9155

4. 6230
4057
9713
1236
9476
9324
1548
7231

5. 7243
2809
4640
2928
3532
2224
7978
9879

6. Copy the table below on a separate sheet. Add across and down, and check the totals by both vertical and horizontal addition.

352	113	4	999	84	
91	246	62	764	603	
817	73	370	382	990	
4	48	454	91	715	
659	294	38	43	341	
487	367	7	207	714	
51	5	581	646	6	

Using algebraic numbers. Have you ever played a game in which a player, who does not succeed in making a certain score, "gets set" or "goes in the hole"? Did you draw a circle around a number, as for example ② to mean "two in the hole"? How might this score be written using plus and minus signs?

In a game Alice and Ruth made these scores. Which girl won?

Alice	*Ruth*	*Alice*	*Ruth*
3	②	3	− 2
5	3	5	3
①	4	− 1	4
④	②	− 4	− 2
1	2	1	2
4	2	4	2

Adding algebraic numbers. The easiest way to add algebraic numbers is to think of the positive numbers as dollars that you have or earn, and of the negative numbers as dollars that you owe or spend.

Or you may use the line on page 100, and think of the direction of the number.

Illustrations.

Find the sum of + 8 and − 10.

+ 8 The + 8 means that I have $8 at first.

− 10 The − 10 means that I spend $10.

− 2 If I had $8 and spent $10, I owe $2.

Find the sum of − 3 and − 5.

− 3 I owe $3.
− 5 I spend $5.
− 8 If I owed $3 and spent $5, I would owe $8.

Find the sum of − 4 and + 9.

− 4 I owe $4.
9 I earn $9 and settle my debt.
5 I have $5 left after paying the debt of $4.

Class Exercises

Add:

1.	+ 5 + 3	+ 4 − 2	+ 7 − 9	− 3 + 8	− 4 + 1
2.	+ 10 − 11	− 7 + 10	− 3 + 3	+ 2 − 6	+ 0 + 5
3.	− 1 − 1	+ 3 0	0 − 3	− 2 + 3	− 9 + 7
4.	3 5	− 3 5	3 − 5	− 3 − 5	− 5 3
5.	− 7 + 3 + 2	+ 4 + 2 − 5	− 3 + 4 − 6	− 8 + 5 + 3	+ 5 − 3 − 6
6.	+ 6 − 16 + 4	− 8 + 5 + 7	− 3 − 2 − 7	+ 4 − 9 − 2	− 7 + 9 − 6
7.	8 − 6 − 3 + 2	− 3 + 5 − 6 + 1	2 5 − 7 − 4	− 6 + 3 + 4 − 2	+ 3 − 7 − 2 − 4

8.	$+3$	-7	$+3$	$+2$	-5
	$+2$	$+2$	-4	-8	-2
	-6	-1	$+1$	$+3$	-4
	$+2$	$+5$	-6	$+1$	$+3$
	-5	-6	$+2$	$+4$	$+5$
	$\underline{-1}$	$\underline{+4}$	$\underline{+4}$	$\underline{-7}$	$\underline{-1}$
9.	8¢	$8c$	4 apples	$4a$	$-6a$
	$\underline{-3}$¢	$\underline{-3c}$	3 apples	$\underline{3a}$	$\underline{+5a}$
10.	− 4 dimes	$-4d$	$6d$	$11d$	$-9d$
	− 8 dimes	$\underline{-8d}$	$\underline{-10d}$	$\underline{3d}$	$\underline{3d}$
11.	$8a$	$2b$	$-3c$	$-5d$	$-8x$
	$\underline{-5a}$	$\underline{-12b}$	$\underline{+2c}$	$\underline{+9d}$	$\underline{+8x}$
12.	$-3F$	$-7g$	$+9h$	$-k$	$+4m$
	$\underline{-4F}$	$\underline{-2g}$	$\underline{-9h}$	$\underline{+2k}$	$\underline{-4m}$
13.	$4a^2$	$-3a^3b$	$7xy^2$	$-5M_1M_2$	$-x_1{}^2$
	$\underline{-6a^2}$	$\underline{+5a^3b}$	$\underline{2xy^2}$	$\underline{-8M_1M_2}$	$\underline{-x_1{}^2}$

14. $2a + 3a + 4a + 6a =$

15. $5b + (-4b) + (-6b) + b =$

16. $(-3c) + 4c + (-7c) + 9c =$

17. $(-2d) + (-5d) + (-8d) + 7d =$

18. $3x + 2x + (-9x) + (-5x) =$

19. $(-5y) + y + 2y + (-4y) =$

20. $(+5a) + (-3a) + (+a) + (-7a) + (+4a) =$

21. $(-2m^2) + (+3m^2) + (-5m^2) + (+4m^2) =$

22. $(+xy) + (-xy) + (-4xy) + (+3xy) =$

23. $(4\pi r^2) + (-3\pi r^2) + (-4\pi r^2) + (+5\pi r^2) =$

24. $(3c) + (+2c) + (-4c) + (+c) + (-5c) =$

25. $8m^2n^3 + (-9m^2n^3) + (-2m^2n^3) + (+m^2n^3) =$

Optional Exercises

26.	$+3a^3b$	$-5bcd$	$2x^2y^3$	$7z^2$
	$-4a^3b$	bcd	$-9x^2y^3$	$2z^2$
	a^3b	$7bcd$	$-x^2y^3$	$-3z^2$
	$\underline{2a^3b}$	$\underline{-6bcd}$	$\underline{+3x^2y^3}$	$\underline{-8z^2}$

27. Count by 2's up from -6 to $+8$, from -5 to $+9$.

28. Count by 3's from -12 to $+9$, from -11 to $+7$.

29. Count by 4's down from $+11$ to -9, from $+10$ to -10.

30. $3x^2 - 5x^2 + 2y^2 - x =$

31. $4xy - 5yz + 2yz - 3xy =$

32. $3a - 2b - 2a + 5b =$

33. $-c + 2c - 3d - 7c =$

34. $x + .3x =$

35. $3a + .05a =$

36. $\frac{1}{2}k + \frac{3}{4}k - \frac{1}{4}k =$

37. $N - .1N =$

38. $5r - .03r + .1r =$

39. $H - 3.46H + 2.1H =$

Honor Work

40.

$$\begin{array}{rrrr} 3.4\,ab & -\ .4\,x^2 & 3.5\,xyz & -5\ \ yz^2 \\ -5.2\,ab & +1.3\,x^2 & -2\ \ xyz & +\ .2\,yz^2 \\ \underline{+3.6\,ab} & \underline{-\ .7\,x^2} & \underline{-5.9\,xyz} & \underline{+3.4\,yz^2} \end{array}$$

41. $5(a + b) - 7(a + b) + 4(a + b) - 8(a + b) =$

42. $-3(x - y) - 6(x - y) + 5(x - y) + (x - y) =$

43. $3(a + b) - 2(x - y) - 4(a + b) + (x - y) =$

44. $a(m + n) + b(m + n) - c(m + n) =$

45. $P(x_1 + x_2) + Q(x_1 + x_2) - R(x_1 + x_2) =$

46.

$$\begin{array}{lllll} Ax & cy & +a^2x & bz & -mxy \\ \underline{Bx} & \underline{-dy} & \underline{-cx} & \underline{z} & \underline{-nxy} \end{array}$$

47. $A(p + q) + B(r - s) - C(p + q) + D(r - s)$

48. $a_1(x_1 + x_2) + a_2(x_1 + x_2) + a_3(x_1 + x_2)$

NOTE: For the addition of polynomials, see page 359.

Algebraic Numbers in Business

1. Principal Hill wished to learn how many books he must order, so he took an inventory. He used plus signs to indicate the number of books a class had more than it needed and minus to tell how many books a class was short. Here is his statement:

Class	*Algebra*	*English*	*Civics*	*Biology*
A	+ 7	− 28	− 18	+ 12
B	− 12	+ 5	− 15	− 32
C	− 10	− 22	− 7	+ 9
D	+ 8	+ 14	+ 16	+ 10

How many of each kind of book must he order?

2. Mr. Jones measured a room of his new house to see how many square feet of plaster he would need. He found that all walls were $9\frac{1}{2}$ ft. high. Here is his record of the dimensions in feet. Can you find the answer for him without multiplying out each number separately? Subtract 60 sq. ft. for doors and windows.

$12 \times 9\frac{1}{2}$
$8 \times 9\frac{1}{2}$
$11 \times 9\frac{1}{2}$
$5 \times 9\frac{1}{2}$
$1 \times 9\frac{1}{2}$
$3 \times 9\frac{1}{2}$

3. Mr. Banker had the following accounts outstanding. Find the total interest due him.

(*a*) $\$387 \times .05\frac{1}{2}$	(*b*) $\$256 \times .04\frac{3}{4}$	(*c*) $\$581 \times .03\frac{1}{2}$
$246 \times .05\frac{1}{2}$	$762 \times .04\frac{3}{4}$	$432 \times .03\frac{1}{2}$
$492 \times .05\frac{1}{2}$	$244 \times .04\frac{3}{4}$	$174 \times .03\frac{1}{2}$
$175 \times .05\frac{1}{2}$	$338 \times .04\frac{3}{4}$	$413 \times .03\frac{1}{2}$

4. Chester Green, a broker, found these net changes each day last week in his stocks:

Stock	*Mon.*	*Tue.*	*Wed.*	*Thu.*	*Fri.*	*Sat.*	*Week*
American Gas..	$+1$	$+\frac{3}{4}$	$-2\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{2}$	$+\frac{3}{4}$	
Standard Coal..	-2	$-\frac{3}{4}$	$+1\frac{1}{2}$	$-\frac{3}{4}$	$+1$	$+\frac{1}{4}$	
United Can....	$-\frac{3}{4}$	$+\frac{1}{2}$	$+1$	$-\frac{1}{2}$	$+1\frac{3}{4}$	-1	

Find the net change in each stock for the week.

5. The Black Coal Company bought this coal during the month of October. Find the total amount due for the month.

Oct. 1...71 T at $\$6\frac{1}{4}$
Oct. 8...83 T at $6\frac{1}{4}$
Oct. 11...65 T at $6\frac{1}{4}$
Oct. 16...58 T at $6\frac{1}{4}$
Oct. 21...93 T at $6\frac{1}{4}$
Oct. 25...70 T at $6\frac{1}{4}$

SIMPLE EQUATIONS

Problem: Smith and Brown buy suits at \$16. At what price must they sell them to make 20% of the selling price?

Let s = the selling price

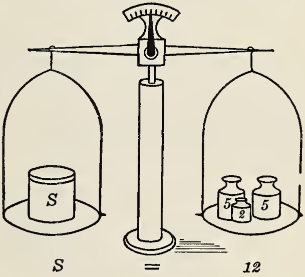
The profit is $\frac{1}{5}$ of the selling price, so

$$\frac{1}{5}s = \text{the profit}$$

If we take the profit from the selling price, the remainder will be the cost, so

$$s - \tfrac{1}{5}s = 16$$

This is an algebraic equation. If we could learn how to ***solve*** such equations, that is, to find the answer to them, we could do many problems that are very difficult or even impossible by arithmetic. Perhaps you can do the problem given here by arithmetic or even get the answer in your head, but there are many problems that you could not do in either of those ways. The algebraic equation is of use in solving such problems.



The equation is a balance. Think of the equation as a balance for weighing. Suppose there are s lbs. of sugar in the left pan and 12 lbs. of weights in the right pan, and that they balance. If I wish to keep them in balance and:

1. If I add 2 lbs. of sugar to the left pan, what must I do to the right pan?

2. If I take 5 lbs. of sugar from the left pan, what must I do to the right pan?

3. If I put 3 times as much sugar in the left pan, what must I do to the right pan?

4. If I leave only $\frac{1}{4}$ as much sugar in the left pan, what must I do to the right pan?

5. If I replace the two 5-lb. weights by a 10-lb. weight, will it remain balanced?

How to solve an equation. You can see that the correct answer to the equation $s - \frac{1}{5}s = 16$ is $s = 20$, for $20 - \frac{1}{5} \times 20$ is 16. Notice that in the answer, $s = 20$, s is all alone on the left side of the equals sign and 20 is on the right side. To solve an equation then, you must get the letter on the left side and have nothing else there. You must get rid of any quantity that is on the wrong side. To do this, you need certain rules, or ***axioms.*** Here is one of them:

Axiom 1. *If the same number is added to both members of an equation, the equation will still balance, or:*

If equals are added to equals, the results are equal.

$$(=\text{s} + =\text{s are } =.)$$

Illustration 1. Solve the equation: $x - 3 = 8$.

Remember we must get the x alone on the left. That means that we must get rid of the -3. To do this, we add 3 to both sides.

$x - 3 = 8$	The given equation.
$3 = 3$	Any quantity equals itself.
$x = 11$	If equals are added to equals, the results are equal.

Check: Substitute 11, the answer, for x in the original equation:

$11 - 3 \mid 8$	A quantity may be substituted for an equal
$8 = 8$	one.

Illustration 2. Solve: $4x = 7 + 3x$

We want the x on the left side of the equals only, so we must first get rid of the $3x$ on the right side. We can do this by adding $-3x$ to both sides.

$4x = 7 + 3x$	The given equation.
$-3x = \quad -3x$	Any quantity equals itself.
$x = 7$	If equals are added to equals, etc.

Check: Substitute 7 for x in the original equation.

4×7	$7 + 3 \times 7$	A quantity may be sub-
28	$7 + 21$	stituted for an equal one
$28 =$	28	(substitution).

Class Exercises

Solve these equations by adding the same quantity to both sides, and give the reason for each step. Check your answers.

1. $x - 4 = 1$
2. $x - 7 = 8$
3. $x - 2 = 5$
4. $x - 3 = -3$
5. $x - 6 = -10$
6. $x + 4 = 9$
7. $x + 7 = 15$
8. $3x = 2x + 4$
9. $4x = 3x + 11$
10. $7x = 6x - 5$
11. $13x = 7 + 12x$
12. $5x = 7 + 4x$
13. $9x = 8 + 8x$
14. $2x - 3 = x$
15. $8x - 9 = 7x$

Optional Exercises

16. $5x - 4 = 4x + 7$
17. $3x + 5 = 2x + 8$
18. $4x - 3 = 3x + 7$
19. $8x - 7 = 7x + 5$
20. $2x - 1 = x + 1$
21. $9x + 7 = 8x - 1$
22. $7x + 13 = 6x + 5$
23. $10x + 11 = 9x + 11$
24. $5x - 3 = 4x - 5$
25. $8x + 2 = 7x - 8$

Honor Work

26. $4x + 3 - 2x - 7 = x$
27. $5x - 9 - 4x + 11 = 0$
28. $2x + 4 + 3x - 6 = 4x$
29. $3.4x - 6.3 = 2.4x$
30. $1.7x + 70 = .7x$
31. $x - a = b$
32. $3x - a = 2x + b$
33. $5x + 3a = 4x - 2a$
34. Solve for C: $C + L = A$
35. Solve for g: $c + g = s$

The Equation in Puzzles

Many of the puzzles that you find in newspapers or that your friends spring on you may be solved by algebra.

Puzzle. Charles said to George: "Think of a number. Now add 8. Now subtract 3. What answer do you get?" George answered, "11." Charles said, "You started with 6." Do you know how Charles got the answer? Well, he wrote:

Let x = the number

$x + 8 - 3 = 11$	The equation made from the problem.
$x + 5 = 11$	Collecting like terms.
$-5 = -5$	Any quantity equals itself.
$x = 6$	If equals are added to equals, etc

What is the number?

1. I am thinking of a number. If I subtract 5 from it and then add 2 to the result, my answer will be 9.

2. If I take 7 away from 6 times a number, I shall have 5 times the number left.

3. If 4 times a number is diminished by 14, the result is 3 times the number.

4. If I multiply a certain number by 4 and then add 8, I shall have 5 times the number.

5. A quart of milk without the bottle sells for 11¢. Katherine returned 3 empty bottles and paid in addition 16¢ to get 2 qts. of milk in 2 bottles. What amount did she get for each bottle that she returned?

6. If I double a certain number and add 3, the result is 11 more than the number.

7. If 3 times a number is increased by 10, the result is 6 more than twice the number.

Thought Question

Paul solved a problem that asked, "In how many years will Ann be twice as old as Mary?" His answer was -2. What does it mean?

Review Exercises

1. Explain the meaning of these: the year -44, the latitude $+41°$, a profit of $-\$400$, the elevation of the Dead Sea above sea level is -1290 ft.

2. Count from -32 to $+43$ by 3's.

3. Find the sum of $4\,hw$, $-2\,hw$, $-7\,hw$, $2\,hw$, and $6\,hw$.

4. Add:

(*a*)	$+8$	(*b*)	$-4\,a^2bc^3$	(*c*)	38452	(*d*)	5×8.37
	-6		a^2bc^3		29614		9×8.37
	$+1$		$2\,a^2bc^3$		7859		3×8.37
	-5		$-4\,a^2bc^3$		76428		7×8.37
	$+3$		$3\,a^2bc^3$		306		6×8.37

5. Express y yds. as feet.

6. If an angle grows larger, in what way does its supplement change?

7. Using the formula $r = \frac{A - p}{pt}$, find r when $A = 442$, $p = 340$, and $t = 6$.

8. Find x: $3x - 4 = 19$.

9. How many cents must Ed pay for s 1¢ stamps and t 3¢ stamps?

10. Using x and y for the numbers, express:
(*a*) The sum of two numbers
(*b*) The difference of two numbers
(*c*) The square of the first number
(*d*) Twice the product of the numbers
(*e*) Three more than the second number
(*f*) The sum of the squares of two numbers

11. Add:
(*a*) $(+5) + (-8)$
(*b*) $(-3) + (-7)$
(*c*) $(-2) + (+5)$
(*d*) $(+2) + (-5) + (+4)$
(*e*) $(-7) + (+2) + (+4)$
(*f*) $(+4) + (-3) + (-1)$

12. By substituting, find whether 3 or 5 is the correct root of the equation $3x + 4 = 7x - 16$.

13. Solve and check: $7x - 13 = 11 + 6x$.

14. Write the equation and solve: If 5 times a certain number is decreased by 13, the result is 4 times the number.

15. Anthony has 7 marbles less than Tom, and together they have 45 marbles. How many has Tom?

16. In $3x + y = 20$, how does y change when x grows larger?

17. Using the formula $C = \frac{5}{9}(F - 32)$, find C when $F = 50$.

18. When two lines cross, what is the name of the pair of opposite angles?

19. Has the normal frequency curve an axis of symmetry? If so, in what direction does it run?

20. What is the average of $+15$, -10, and $+7$? Of $+4$, -7, and $+3$?

21. Explain why subscripts are sometimes employed in formulas.

22. Make a formula for the distance an automobile will go in t hrs. at 40 mi. an hr.

23. What does -2 in the net change of a stock-market report mean?

Make graphs suitable for these exercises:

24. Paul walks 3 mi. an hr. Make a graph showing the distance d he will go in t hrs. from $t = 0$ to $t = 8$.

25. The average temperature for June in these cities is: Albany, 68°; Atlanta, 76°; Bismarck, 64°; Dallas, 81°; Miami, 80°; and San Francisco, 58°.

26. The population of Michigan increased as follows:

1810	*1830*	*1850*	*1870*	*1890*	*1910*	*1930*
5000	32,000	398,000	1,184,000	2,094,000	2,810,000	4,842,000

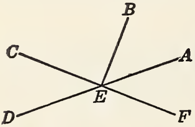
27. Draw a small angle, and construct one 5 times as large.

28. In this figure, read with 3 letters:

(*a*) An acute angle (*c*) An obtuse angle

(*b*) A right angle (*d*) A straight angle

29. Is any angle supplementary to AEB?



30. Is any angle vertical to AEB?

31. An angle has 78°. How many degrees are there in its supplement? In its vertical angle?

32. Is a fish symmetrical? Is your image in a mirror symmetrical to you?

33. Write a formula for the perimeter of:

(*a*) A square whose side is $3a + 4$

(*b*) A rectangle whose width is w and whose length is $2w + 7$

(*c*) A triangle whose sides are a, b, and c

34. Using the formula $T = 2e^2 + 4he$, find T when $e = 8$ and $h = 5$.

35. Make a frequency polygon to illustrate these results: In a test 1 pupil received 40%; 3, 50%; 5, 60%; 8, 70%; 6, 80%; 4, 90%; and 2, 100%.

36. Find the average, median, and mode: In a class contribution 7 pupils gave 5¢, 9 gave 10¢, 12 gave 15¢, and 2 gave 25¢.

37. A watch ticked 80 times from the time a flash of lightning was seen until the thunder was heard. How many miles away was the lightning?

38. At what price must a clothier mark a suit so as to give a discount of 40% and sell it for $18.60?

$$\text{Formula: } m = \frac{s}{1 - d}$$

39. Sarah sat 5 ft. from the fulcrum of a teeter board. Robert, who weighs 90 lbs., found that he balanced her when he sat 6 ft. from the support. What was Sarah's weight?

$$\text{Formula: } w_1 = \frac{d_2 w_2}{d_1}$$

40. An elasticity formula from science is $M = \frac{Lf}{ea}$. Find M if $L = 144$, $f = 25$, $e = .001$, and $a = 3$.

Test on Addition

1. $\begin{array}{r} 5 \\ -3 \\ \hline \end{array}$

2. $\begin{array}{r} -6 \\ 3 \\ \hline \end{array}$

3. $\begin{array}{r} -4 \\ -5 \\ \hline \end{array}$

4. $\begin{array}{r} -6 \\ 9 \\ \hline \end{array}$

5. $\begin{array}{r} 3 \\ -3 \\ \hline \end{array}$

6. $\begin{array}{r} 3x \\ 5x \\ \hline \end{array}$

7. $\begin{array}{r} -4x \\ -3x \\ \hline \end{array}$

8. $\begin{array}{r} 3ab \\ -6ab \\ \hline \end{array}$

9. $\begin{array}{r} -M \\ -3M \\ \hline \end{array}$

10. $\begin{array}{r} -2h \\ 7h \\ \hline \end{array}$

11. $\begin{array}{r} 3x^2y^3 \\ 4x^2y^3 \\ \hline \end{array}$

12. $\begin{array}{r} -8k^4 \\ -k^4 \\ \hline \end{array}$

13. $\begin{array}{r} 3m_1m_2 \\ -m_1m_2 \\ \hline \end{array}$

14. $\begin{array}{r} -9pv^3 \\ 10pv^3 \\ \hline \end{array}$

15. $\begin{array}{r} -10hw \\ 10hw \\ \hline \end{array}$

16. $\begin{array}{r} 2.4k \\ 3.7k \\ \hline \end{array}$

17. $\begin{array}{r} 5.8h^2 \\ -3.5h^2 \\ \hline \end{array}$

18. $\begin{array}{r} \frac{1}{2}m \\ -m \\ \hline \end{array}$

19. $\begin{array}{r} 2\frac{1}{2}r^3 \\ -3r^3 \\ \hline \end{array}$

20. $\begin{array}{r} -3\frac{1}{3}B \\ 2\frac{1}{6}B \\ \hline \end{array}$

Test on Equations with Axioms

Solve these equations, and give a reason for each step. Check your answers.

1. $x - 5 = 4$
2. $x - 7 = 2$
3. $x - 9 = -3$
4. $x + 1 = -4$
5. $x - 5 = -5$
6. $x - 3.4 = 4.6$
7. $x + 1.3 = 8.3$
8. $x - 5.6 = -2.6$
9. $x + 2.9 = 2.9$
10. $x - 5.3 = -8.3$
11. $3x - 4 = 2x + 7$
12. $5x - 1 = 4x + 9$
13. $4x - 3 = 8 + 3x$
14. $9x + 6 = 10 + 8x$
15. $5x + 8 = 4x - 2$

Chapter 6

SUBTRACTION

Review of arithmetic. *Subtract:*

1. $\begin{array}{r}38452\\ \underline{21965}\end{array}$	**4.** $\begin{array}{r}54390\\ \underline{45077}\end{array}$	**7.** $\begin{array}{r}5142\\ \underline{3746}\end{array}$	**10.** $\begin{array}{r}4.9852\\ \underline{4.2769}\end{array}$
2. $\begin{array}{r}91002\\ \underline{73874}\end{array}$	**5.** $\begin{array}{r}6452\\ \underline{5849}\end{array}$	**8.** $\begin{array}{r}9780\\ \underline{9674}\end{array}$	**11.** $\begin{array}{r}5.0000\\ \underline{2.8635}\end{array}$
3. $\begin{array}{r}12764\\ \underline{875}\end{array}$	**6.** $\begin{array}{r}8763\\ \underline{1784}\end{array}$	**9.** $\begin{array}{r}31.768\\ \underline{24.817}\end{array}$	**12.** $\begin{array}{r}87\\ \underline{9.635}\end{array}$

13. Beginning at 100, count down by 3's, by 4's, by 6's, by 7's, by 8's, by 9's.

Making change. When you give a dollar bill in paying a debt of 32¢, the merchant makes change by addition. He hands you 3 pennies, a nickel, a dime, and a half dollar, saying as he does so, "32¢, 35¢, 40¢, 50¢, $1."

Make change from a dollar bill for each of these amounts:

14. 48¢
15. 89¢
16. 15¢
17. 72¢
18. 60¢
19. 18¢
20. 42¢
21. 37¢

ALGEBRAIC SUBTRACTION

George Washington was born in 1732 and died in 1799. How many years did he live? What did you do to the numbers 1799 and 1732 to get your answer? Can you always find the number of years a person lived by subtracting the date of his birth from that of his death?

Augustus Caesar was born in − 63 and died in + 14. How many years did he live? Evidently he lived 63 yrs. before Christ and 14 yrs. afterwards, or 77 yrs. Can we get the answer 77 by subtracting − 63 from + 14?

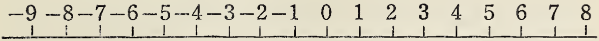
Subtraction as the opposite of addition.

Example 1. Find the value of $+8-(+3)$.

Just as the clerk makes change by addition, so we can subtract 3 from 8 by asking ourselves what number we must add to 3 to make 8. If the numbers are laid off on a scale going in both directions from 0, we can count the number of spaces from 3 to 8. The answer is 5. We counted to the right. Is the answer positive or negative? Let us see.

Add $+2$ to $+5$. Do you count to the left or to the right when you add a positive number to 5?

Add -3 to $+5$. Do you count to the left or to the right when you add a negative number to 5?



To add a positive number, count to the right; to add a negative number, count to the left.

Example 2. Find the value of $-7-(+4)$. This is the same as asking what number we must add to 4 to make -7. So starting at $+4$, we find that -7 is 11 spaces to the left. The answer then is -11.

Exercises

What number must be added to the first number to give the second?

1. 1, 6
2. 2, 8
3. 4, 0
4. -3, 5
5. -7, 1
6. -5, 8
7. 2, -5
8. 5, 2
9. 4, -7
10. -5, -3
11. -2, -8
12. 0, -6

Investigating subtraction.

1. Subtract: $\begin{array}{r}7\\ \underline{3}\end{array}$ *Think:* What number must I add to 3 to make 7?

2. Subtract: $\begin{array}{r}6\\ \underline{-4}\end{array}$ *Think:* What number must I add to -4 to make 6?

3. Subtract: $\begin{array}{r} -2 \\ \underline{-5} \end{array}$ *Think:* What number must I add to −5 to make −2?

4. Subtract: $\begin{array}{r} -7 \\ \underline{+3} \end{array}$ *Think:* What number must I add to +3 to make −7?

If you did these four exercises correctly, your answers were:

$$\begin{array}{rrrr} 7 & 6 & -2 & -7 \\ +3 & -4 & -5 & +3 \\ \hline 4 & 10 & 3 & -10 \end{array}$$

Look at these addition exercises:

$$\begin{array}{rrrr} 7 & 6 & -2 & -7 \\ -3 & +4 & +5 & -3 \\ \hline 4 & 10 & 3 & -10 \end{array}$$

Compare the answers to these addition exercises with the answers to the subtraction exercises above them. Are they the same? Now compare the upper numbers (minuend). Are they the same? Finally compare the numbers subtracted (subtrahend). Are they the same? How then can we change subtraction exercises into exercises in addition?

Rule for subtraction. Change the sign of the number to be subtracted, the lower number, and proceed as in addition.

Of course, you will make this change mentally only, and not write it on your paper.

Class Exercises

Subtract. Then check by addition.

1. $\begin{array}{rrrrrr} +6 & +5 & -7 & -4 & -8 & 8 \\ +2 & -3 & -1 & -9 & +2 & 6 \\ \hline \end{array}$

2. $\begin{array}{rrrrrr} 5 & 6 & -4 & -2 & 0 & 4 \\ 9 & -3 & 7 & -1 & -1 & -3 \\ \hline \end{array}$

3. $\begin{array}{rrrrrr} 5 & -7 & -8 & 0 & -6 & -7 \\ -1 & 2 & -3 & 5 & 0 & 5 \\ \hline \end{array}$

4. $$\begin{array}{r} -4 \\ -6 \\ \hline \end{array} \qquad \begin{array}{r} -7 \\ -2 \\ \hline \end{array} \qquad \begin{array}{r} -5 \\ -4 \\ \hline \end{array} \qquad \begin{array}{r} -2 \\ +6 \\ \hline \end{array} \qquad \begin{array}{r} -1 \\ 3 \\ \hline \end{array} \qquad \begin{array}{r} -0 \\ -2 \\ \hline \end{array}$$

5. $$\begin{array}{r} -2 \\ -2 \\ \hline \end{array} \qquad \begin{array}{r} +5 \\ +5 \\ \hline \end{array} \qquad \begin{array}{r} 8 \\ -3 \\ \hline \end{array} \qquad \begin{array}{r} -4 \\ +9 \\ \hline \end{array} \qquad \begin{array}{r} -2 \\ -6 \\ \hline \end{array} \qquad \begin{array}{r} +4 \\ -4 \\ \hline \end{array}$$

6. $$\begin{array}{r} 3 \\ 0 \\ \hline \end{array} \qquad \begin{array}{r} 0 \\ 4 \\ \hline \end{array} \qquad \begin{array}{r} -5 \\ -1 \\ \hline \end{array} \qquad \begin{array}{r} -6 \\ -2 \\ \hline \end{array} \qquad \begin{array}{r} +0 \\ -0 \\ \hline \end{array} \qquad \begin{array}{r} -4 \\ 5 \\ \hline \end{array}$$

7. $$\begin{array}{r} -3a \\ -5a \\ \hline \end{array} \qquad \begin{array}{r} -2b \\ +4b \\ \hline \end{array} \qquad \begin{array}{r} 2c^2 \\ 6c^2 \\ \hline \end{array} \qquad \begin{array}{r} 5d \\ 4d \\ \hline \end{array} \qquad \begin{array}{r} 3e^3 \\ -3e^3 \\ \hline \end{array}$$

8. $$\begin{array}{r} 7FG \\ -2FG \\ \hline \end{array} \qquad \begin{array}{r} 4w^2x \\ 7w^2x \\ \hline \end{array} \qquad \begin{array}{r} -3m^3n^3 \\ -4m^3n^3 \\ \hline \end{array} \qquad \begin{array}{l} 0 \\ 2m_1m_2 \\ \hline \end{array} \qquad \begin{array}{r} A^2 \\ -2A^2 \\ \hline \end{array}$$

9. $$\begin{array}{l} xyz \\ xyz \\ \hline \end{array} \qquad \begin{array}{r} 0 \\ -3yz^4 \\ \hline \end{array} \qquad \begin{array}{l} 5y^3 \\ 2y^3 \\ \hline \end{array} \qquad \begin{array}{r} 4r_1{}^2r_2{}^3 \\ r_1{}^2r_2{}^3 \\ \hline \end{array} \qquad \begin{array}{r} 2\pi r^2h \\ \pi r^2h \\ \hline \end{array}$$

10. $+(3x)-(+2x)-(-5x)$

11. $+(-5a)-(-7a)+(-3a)$

12. $-(+2r)-(-3r)-(-r)$

13. $-(-m^2)+(-2m^2)-(+3m^2)$

14. $+(E^3)-(2E^3)-(-5E^3)$

15. $-(4w_1w_2)+(-2w_1w_2)-(-w_1w_2)$

Optional Exercises

16. $$\begin{array}{r} 3.8x \\ -2.2x \\ \hline \end{array} \qquad \begin{array}{r} 7.34y \\ 3.29y \\ \hline \end{array} \qquad \begin{array}{r} 5.4c \\ 2.77c \\ \hline \end{array} \qquad \begin{array}{r} ab \\ .2ab \\ \hline \end{array} \qquad \begin{array}{r} .9z \\ z \\ \hline \end{array}$$

17. $$\begin{array}{l} 3\frac{1}{3}M \\ 2\frac{1}{2}M \\ \hline \end{array} \qquad \begin{array}{r} 2\sqrt{x} \\ \sqrt{x} \\ \hline \end{array} \qquad \begin{array}{r} -5(a+b) \\ -2(a+b) \\ \hline \end{array} \qquad \begin{array}{r} 3\sqrt{x-y} \\ -4\sqrt{x-y} \\ \hline \end{array}$$

18. What must I add to $5x$ to make $-3x$? To make 0? To make $4b$?

19. How much greater is $-5a$ than $-9a$? Than $8m$?

Honor Work

20. $$\begin{array}{l} a \\ b \\ \hline \end{array} \qquad \begin{array}{l} cy \\ dy \\ \hline \end{array} \qquad \begin{array}{r} ax \\ x \\ \hline \end{array} \qquad \begin{array}{r} bx^2 \\ 2x^2 \\ \hline \end{array} \qquad \begin{array}{r} M_1(a+b) \\ M_2(a+b) \\ \hline \end{array}$$

21. $$\begin{array}{l} 2amn \\ 2bmn \\ \hline \end{array} \qquad \begin{array}{r} 3x^a \\ -2x^a \\ \hline \end{array} \qquad \begin{array}{r} a\sqrt{x+y} \\ \sqrt{x+y} \\ \hline \end{array} \qquad \begin{array}{l} abx \\ acx \\ \hline \end{array} \qquad \begin{array}{l} aXx \\ bXx \\ \hline \end{array}$$

22. If the value of x begins at -5 and grows larger, does it move toward 0 or away from 0?

23. In subtraction, how does the answer change: (*a*) If the upper number increases, but the lower number remains unchanged? (*b*) If the lower number increases, but the upper number remains unchanged? (*c*) If both numbers grow larger by the same amount?

Applied Problems

24. At 6 A.M. the temperature was − 8° and at noon it was + 18°. How many degrees did it rise?

25. A ship traveled north from latitude − 42° to latitude − 14°. How many degrees north did it travel?

26. In a game Fred has − 13 points and Robert has + 8. How many points must Fred win to overtake Robert?

27. Last August Katherine had − $3 and now she has + $7. How many dollars did she save?

28. How many years did each of these people live?

	Born	*Died*
Lincoln	+ 1809	+ 1865
Charlemagne	+ 742	+ 814
Alexander the Great	− 356	− 323
Constantine	+ 272	+ 337
Jesus Christ	− 4	+ 29
Julius Caesar	− 100	− 44
Ovid	− 43	+ 18
Livy	− 59	+ 17

29. How many years elapsed from the founding of the Roman Empire in − 753 to its end in + 476?

30. How long did the Alexandrine age of literature last if it began about − 323 and ended about + 100?

31. Euclid lived about − 300 and Descartes in + 1600. How many years passed from the time of Euclid to that of Descartes?

32. The boiling point of liquid hydrogen is − 253° C.; of liquid oxygen, − 182° C.; and of water, + 100° C. Which is warmer and how much warmer when boiling: (*a*) hydrogen or oxygen? (*b*) Hydrogen or water? (*c*) Oxygen or water?

33. Water freezes at 0° C. Is boiling oxygen warmer than ice?

NOTE: For the subtraction of polynomials, see page 360.

EQUATIONS

We learned that we could change the form of an equation by adding the same amount to both members. Sometimes it was necessary to add a negative amount. Instead of this, we could have subtracted a positive amount and used the axiom:

Axiom 2. *If the same quantity is subtracted from both members of an equation, the equation will still balance, or:*

If equals are subtracted from equals, the results are equal.

(=s − =s are =.)

Illustration. Solve for x: $3x + 7 = 2x + 12$

Here we must get rid of the 7 from the left member and the $2x$ from the right member.

$3x + 7 = 2x + 12$	The original equation.
$\underline{2x + 7 = 2x + 7}$	Any quantity equals itself.
$x = 5$	If equals are subtracted from equals, the results are equal.
Check: $3 \times 5 + 7 \mid 2 \times 5 + 12$	Substitution.
$15 + 7 \mid 10 + 12$	
$22 = 22$	

Exercises

Solve these equations giving a reason for each step. Check.

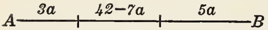
1. $x + 3 = 7$
2. $y + 10 = 15$
3. $v + 8 = 4$
4. $t + 5 = -3$
5. $m_1 + 2\frac{1}{2} = 4\frac{1}{2}$
6. $3R + 5 = 2R + 8$
7. $5h + 7 = 4h - 3$
8. $2s - 5 = s + 5$
9. $4w + 8 = 3w + 8$
10. $8E + 1.6 = 7E + 3.6$

Problems

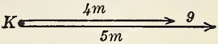
11. One number is $3x$, and another is $12 - 2x$. Their sum is 16. Find x and both numbers.

12. One number is $2n$; a second number is $3n$; and a third is $10 - 4n$. Their sum is 12. Find n and all three numbers.

13. AB is 47 in. long and is cut into 3 parts, $3a$, $42 - 7a$, and $5a$. Find the value of a and the length of each segment of AB.



14. Paul and John started from K, and both traveled east. Paul went $5m$ mi., and John went $4m$ mi. They were then 9 mi. apart. How far did each travel?



Thought Questions

If x is growing larger, how is $15 - x$ changing? How large must x grow so that $15 - x$ will become 6?

PARENTHESES

Where have you already used the parenthesis? Of what use was it in multiplication? Have you used it in addition or subtraction?

What is the value of $9 - (5 - 2)$? If this parenthesis were simply left out and we wrote it $9 - 5 - 2$, would we get the same answer?

Parentheses are used to indicate that a certain operation is to be performed on the enclosed expression as a whole. To remove the parenthesis, simply perform this operation. They are most often used to show that the enclosed expression is to be multiplied by some number or is to be subtracted from some number. Other signs, the bracket [ ], the brace { }, and the vinculum $\overline{\quad\quad}$ are sometimes used in the same way as the parenthesis.

The parenthesis in addition and subtraction.

Illustration 1. $2a - b + (c - 3d)$ means that $c - 3d$ is to be added to $2a - b$.

Adding:

$$\begin{array}{r} 2a - b\phantom{{}+c-3d} \\ c - 3d \\ \hline 2a - b + c - 3d \end{array}$$

Since in addition we do not change the sign of any term, the parenthesis is of no value here and may be dropped without changing the expression.

Check by substituting a number for each letter in the original expression and in your answer.

Illustration 2. $2x^3 - x^2 - (3x - 4)$ means that $3x - 4$ is to be subtracted from $2x^3 - x^2$.

$$\text{Subtracting:}\quad \begin{array}{r} 2x^3 - x^2 \phantom{{}-3x+4} \\ 3x - 4 \\ \hline 2x^3 - x^2 - 3x + 4 \end{array}$$

Since in subtraction we change the signs of all terms subtracted, the parenthesis may be omitted provided we change the signs of all terms that were inside it.

Illustration 3. $2x - 3 - (-x - 4) + (5 - x)$. We change the signs of all terms that were in the parenthesis preceded by the minus sign, but do not change those in the parenthesis preceded by the plus sign. After the parentheses are removed, when there are like terms, we collect them into a single term.

$$\begin{aligned} &2x - 3 - (-x - 4) + (5 - x) \\ =\ &2x - 3 \quad + x + 4 + 5 - x \\ =\ &2x + 6 \end{aligned}$$

Class Exercises

Remove parentheses, and collect like terms:

1. $3a - b + (a - b)$
2. $X + 4 - (2 - X)$
3. $5b - 4 - (2b + 3)$
4. $3\pi + (3 + \pi)$
5. $3T - (T - 4)$
6. $-2m - (-3m - 4)$
7. $x^2 - (2x + 4)$
8. $m + (2m - 3)$
9. $k - (-1 - k)$
10. $7y + 3z + (-2y - 5z)$
11. $A - B - (-A + B)$
12. $5 - c - (c - 2)$

Optional Exercises

13. $(2a - 3c) + (2c - 3a)$
14. $(a_1 + a_2) - (2a_1 + a_2)$
15. $-(3x - 4y) + (5x - y)$
16. $-(-z - 2) - (-4 - 5z)$
17. $(2r + S) + (-2r - S)$
18. $-(2f + 3g) + (f - 3g)$
19. $2x - (3x^2 - 2x + 4)$
20. $-4 + (m^2 - m + 4)$
21. $3x^2 - (x^2 + x - 1) - x$

22. $-(-x^2 + x - 3) - 2x^2 + 1$

23. $3W + (v - 3W) + 4 - v$

24. $5m - (3 - 2m) - (4m - 2)$

Honor Work

25. $a + 2b + [-5a + b - (3a - 2b) - 2b] + a$

26. $3x^2 - 4 - [2x - (x^2 - x) + (3x - 5) + 4x^2] - 2$

27. $3y^2 - (2y + 6) - [-(5y + 3) + (-4 + 2y)]$

28. $[-\{3 - (2 + 4x)\} + 1] - [\{-4 + (2x - 7)\} - 6]$

29. $-(A - B) - \{[-(2A - B) - (-3A - 5B)] - B\}$

30. $x - \{-4 + [2x + 5 - \overline{3 - 7x} + 4] - (x + 1) + 3\}$

Inserting parentheses. If we enclose terms in a parenthesis and then remove the parenthesis again, we ought to get back the same expression that we had at first. Consequently when we enclose terms in a parenthesis, we change the signs of all the terms enclosed if the expression is preceded by a minus sign, but we do not change the signs if it is preceded by a plus sign.

Exercises

Inclose the last two terms in a parenthesis preceded by (a) a plus sign; (b) a minus sign:

1. $a + b + c$

2. $M - N - R$

3. $5x - y + 7z$

4. $ar - bs - ct$

5. $mn + np - pq$

6. $3x^2 - 4xy + 2y^2$

7. $cd + df - gh$

8. $5a^2 + 3a + 2$

9. $a^2 - b^2 + c^2$

10. $Ax + By - Cz$

EQUATIONS CONTAINING PARENTHESES

Illustration. Solve and check: $5x - (4x - 8) = 18$

$$\begin{aligned} 5x - (4x - 8) &= 18 \\ 5x - 4x + 8 &= 18 \\ x + 8 &= 18 \\ 8 &= 8 \\ x &= 10 \end{aligned}$$

Check:

$$\begin{array}{r|l} 50 - (40 - 8) & 18 \\ 50 - 32 & 18 \end{array}$$

$$18 = 18$$

Exercises

Solve and check:

1. $3x - (2x - 1) = 4$
2. $9y - (6 + 8y) = 2$
3. $4k + (8 - 3k) = 10$
4. $5z - (2 + 4z) = 8$
5. $2m + (3 - m) = 4$
6. $7r - (6r - 5) = 9$
7. $(3 - x) + 2x = 1$
8. $5y - (4y - 6) = 0$
9. $4a + (3 - 3a) = -7$
10. $4x = 7 - (2 - 3x)$
11. $(3 - 2x) + (3x - 1) = 0$
12. $5 - (2y - 3) = -(4 + 3y)$
13. $4 = 2x - (x - 2)$
14. $3y + 1 + (5 - 2y) = -8$
15. $(3x - 4) + (2 - 2x) = 4$
16. $5x - (4 - 3x) = 5 + 7x$
17. $2y + (3y + 9) - 4y = 0$
18. $3 = (5x - 2) - (3 + 4x)$
19. $9 - (2x - 1) = -3 - (4 + 3x)$
20. $(7 - 5x) - 3 = 2 + (1 - 6x)$

Algebraic Expression

1. Which represents 4 more than k: $4k$ or $k + 4$?

2. If y is a number, what does $3y$ mean? What does $y + 5$ mean?

3. Write a number 4 larger than x; 3 times as large as x.

4. Two numbers differ by 5. The smaller is 7. What is the larger?

5. Two numbers differ by 5. The smaller is a. What is the larger?

6. Two numbers differ by d. The smaller is x. What is the larger?

7. By how much does 10 exceed n? y exceed 7? y exceed n?

8. What number exceeds 7 by x? Exceeds p by q?

9. Using n to represent a number, write:

(*a*) A number increased by 4
(*b*) A number diminished by 7
(*c*) Twice a number
(*d*) 6 less than a number

10. The difference of two numbers is y. The larger is 8. Find the smaller.

11. The sum of two numbers is 23. The smaller is 8. Find the larger.

12. The sum of two numbers is 12. The smaller is x. Find the larger.

13. The sum of two numbers is s. The smaller is x. Find the larger.

14. John had m marbles. He lost 9. How many has he left?

15. The smaller of two numbers is x. The larger is 5 times the smaller.

(*a*) Represent the larger.

(*b*) Represent their sum, their difference, their product.

(*c*) Write a number 8 smaller than the larger number, y smaller than the larger number, z larger than the smaller number.

16. Gladys had c¢. She earns 20¢ more. How much has she now?

17. If k is divided into 7 equal parts, what is one part?

18. If 20 is divided into n equal parts, what is one part?

19. If k is divided into n equal parts, what is one part?

20. Separate n into 2 parts so that the smaller is y.

21. X is one part of 15. X is increasing. How is the other part changing?

22. A number exceeds r by 7. If r is increasing, how is the other number changing?

Algebraic Shorthand

Translate the following sentences into algebra, and solve:

1. Six more than a number is 11.

2. When a number is increased by 8, the sum is 15.

3. In 5 yrs., Robert's age will be 16.

4. If Dorothy had 24¢ more, she would have $1.

5. If I add 8 to a number, the result is twice the number.

6. Twice a number increased by 12 equals the number increased by 17.

7. Five times a number is 4 more than 4 times the number.

8. If I subtract 7 from 3 times a number, the result will be 6 more than twice the number.

9. Three times a number exceeds 12 as much as twice the number exceeds 7.

ALGEBRAIC NUMBERS IN MUSIC (Optional)

Have you ever studied music? Do you remember what a nuisance it was to memorize all the keys? Can you tell now what key has 3 sharps or what key has 4 flats? Would you like to learn an easy way by which you could figure it out without memorizing? Here is the rule.

Every time you raise the key a whole tone, add two sharps. For example, D is 1 tone higher than C. So the key of D should have 2 sharps more than the key of C has. Similarly the key of E has 2 sharps more than the key of D.

What then should you do every time you lower the key a whole tone?

When the number of sharps becomes greater than 7, decrease it by 7. This lowers the key $\frac{1}{2}$ tone.

C and G are good starting points, as C has no sharps and G has 1.

JOHANN SEBASTIAN BACH
(1685–1750)

Famous German composer whose works influenced profoundly the course of modern music.

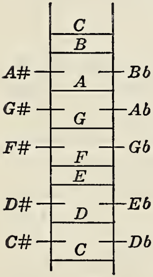


Illustration. How many sharps has A?

Since A is one tone higher than G, add 2 sharps. Then we have $1 + 2 = 3$. A has 3 sharps.

Exercises

1. If G has 1 sharp, how many has B? C♯? (♯ is the symbol for sharp, and ♭ is the symbol for flat).

2. If B has 5 sharps, how many has A? G? F? E♭? D♭? What do we mean by a negative sharp?

3. If C has 0 sharps, how many has D? E? F♯? Or going down from the upper C, how many has B♭? A♭? G♭?

On the ladder, notice that the intervals from E to F and from B to C are only half tones. Consequently, for example, one whole tone down from C gives us B♭, not just B.

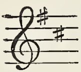
4. How many whole tones is it from D to F♯? If D has 2 sharps, how many has F♯?

5. How many whole tones is it from upper C down to G♭? How many flats (negative sharps) has G♭?

Name each of these keys counting from C when the number of sharps or flats is even, and from G when that number is odd.

From C:

6.



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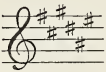

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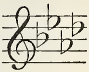

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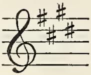

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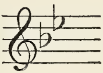
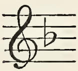


From G:

11.



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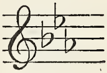

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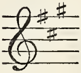

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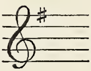

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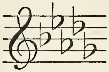


16. If F has 1 flat, what has A?

17. Using G and C as starting points, write in order the key having 1 sharp, 2 sharps, 3 sharps, 4 sharps, 5 sharps, 6 sharps, 1 flat, 2 flats, 3 flats, 4 flats, 5 flats, and 6 flats.

When we lower the key $\frac{1}{2}$ tone, we remove a sharp from each of the 7 notes in the "octave"; that is, we subtract 7 sharps. For example, if A has 3 sharps, A♭ will have $3 - 7$ or -4 sharps or 4 flats.

18. How many flats has:

(*a*) D♭ if D has 2 sharps?
(*b*) E♭ if E has 4 sharps?
(*c*) F if F♯ has 6 sharps?
(*d*) C♭ if C has 0 sharps?

19. How many sharps has:

(*a*) A if A♭ has 4 flats?
(*b*) B if B♭ has 2 flats?

Review Exercises

1. Make a circle graph to show these facts: Dry wood contains 50% carbon, 6% hydrogen, 41% oxygen, 3% nitrogen and ash.

2. From the sum of 8 and -11, subtract -5.

3. Last year Mr. Strong's balance was $-\$800$. This year it is $+\$1300$. How much did he save?

4. Mt. Everest is $+29{,}141$ ft. high. The deepest point in the Pacific Ocean is $-35{,}400$ ft. How far above this point is the top of Mt. Everest?

5. What number diminished by 5 equals -3?

6. How much greater is -2 than -9? 3 than -4?

7. Find the other number if the sum of two numbers is:

(*a*) 15 and one of them is 7
(*b*) 12 and one of them is -4
(*c*) 0 and one of them is 5
(*d*) 8 and one of them is 13
(*e*) -8 and one of them is -12
(*f*) 0 and one of them is -6
(*g*) -2 and one of them is 1
(*h*) -10 and one of them is -6
(*i*) 3 and one of them is 0

8. Solve for x and check: $3x + 7 = 2 + 2x$

9. Show by substituting whether $x = 2$ or $x = 3$ is the correct answer to $x^3 - 6x^2 + 10x = 3$.

10. If an airplane travels m mi. an hr., how many hours will it require to travel 500 mi.?

11. Sarah had 90¢. She bought p pencils at c¢ each. How much had she left?

12. An angle has $x - 30$ degrees. How many degrees are there in

(*a*) Its supplement? (*b*) Its vertical angle?

13. An obtuse angle has $d°$. By how much does it exceed a right angle?

14. How many degrees are there in r right angles? In s straight angles?

15. The sum of the angles of a polygon (a figure made of straight lines enclosing a part of the plane) is found by using the formula $S = (n - 2)180°$ where n is the number of sides.

Find the number of degrees in the sum of the angles of a polygon having:

(*a*) 3 sides (*c*) 6 sides (*e*) 102 sides
(*b*) 4 sides (*d*) 12 sides (*f*) 2 sides

(*g*) When the number of sides increases, does the sum of the angles increase or decrease?

(*h*) When the number of sides is doubled, is the sum of the angles doubled?

16. Make a graph of the formula $S = (n - 2)180$ for values of n from 2 to 10.

17. Robert is y yrs. old. How old was he k yrs. ago? How old will he be in t yrs.?

18. Mrs. Heap bought 2 doz. buttons at d¢ a doz. and 5 yds. of cloth at $4\,d$¢ a yd. What was her total bill?

19. Translate into algebra:

(*a*) Twice a number increased by 10 is 5 less than 6 times the number.

(*b*) Three times a number is as much above 14 as the number is below 14.

(*c*) If 4 times a number is decreased by 9, the result is still 1 more than the number.

(*d*) If I multiply a number by 3 and then increase the product by 5, the result is 9 less than 5 times the number.

20. In a football game Georgia Tech gained 8 yds., − 3 yds., − 1 yd., and 7 yds. Did they make "first down" (10 yds.)?

21. Construct an angle of $22\frac{1}{2}°$.

22. Subtract:

$$\begin{array}{llllllllll} (a) & -5 & (b) & 7\,a^2 & (c) & -13\,m^2n^3 & (d) & 3\,R^2 & (e) & -\ 4\,v \\ & \underline{-3} & & \underline{-\ a^2} & & \underline{8\,m^2n^3} & & \underline{8\,R^2} & & \underline{-11\,v} \end{array}$$

23. Remove the parentheses, and collect terms:

(*a*) $3\,x - (4 + x) + (3\,x - 6)$
(*b*) $2\,a - 5 + (3 - a) - (-4\,a + 7)$

24. A problem asked "How much heavier is Sarah than Katherine?" Dorothy's answer was − 10 lbs. What does it mean?

25. An airship with its passengers weighs − 1500 lbs. After 10 passengers, averaging 150 lbs. each get out, what does it weigh?

Test in Removing Parentheses

Remove the parentheses, and collect terms:

1. $5x + (x - 3)$
2. $7m - (2m + 8)$
3. $2a - (a - 4)$
4. $3b + (7 - b)$
5. $8h - (3 - 2h)$
6. $3r - (-2r + 4)$
7. $4c - (c - 3) + 3c + 4$
8. $h + (3 - h) - (2h - 5)$
9. $5A - (A + 1) - (1 - A)$
10. $-(x + 4) + 3x - (2x - 5)$
11. $p - q - (p - q) + 3$
12. $12 - (-7 + x) + (3x + 2)$

Test in Solving Equations

Solve, giving a reason for each step, and check:

1. $x + 4 = 9$
2. $x + 7 = 13$
3. $x - 3 = 5$
4. $x - 7 = 4$
5. $x + 5 = 2$
6. $x - 8 = -4$
7. $x - 2 = -5$
8. $x + 6 = 6$
9. $4x + 10 = 3x$
10. $6x - 7 = 5x$
11. $7m + 12 = 20 + 6m$
12. $3k - 5 = 2k + 1$
13. $3h + 15 = 10 + 2h$
14. $5 - v = 8 - 2v$
15. $8 - 4n = 8 - 5n$
16. $3i - 2.7 = 1.3 + 2i$
17. $5w + 8.72 = 4w + 9.22$
18. $3.8t - 7 = 4 + 2.8t$
19. $5.97 - .2y = 8.97 + .8y$
20. $.32x - 1.2 = 8.8 - .68x$

Chapter 7

MULTIPLICATION AND DIVISION

MULTIPLICATION

Review of arithmetic.

1. Add the column of figures on the right.

2. Can you discover a short cut for this addition? Check your answer using this short cut.

3. How would you multiply a number by 10? By 100? By 1000?

4. Find the product: (*a*) 3849 × 10; (*b*) 926 × 100; (*c*) 386 × 1000.

$$\begin{array}{r}48736\\48736\\48736\\48736\\48736\\48736\\48736\\48736\\48736\\\underline{48736}\end{array}$$

Find these products:

5. 382 × 8 **7.** 5714 × 387 **9.** 41567 × 3863
6. 471 × 9 **8.** 1497 × 429 **10.** 86254 × 7421

11. When you multiply a fraction by a whole number, do you multiply the numerator only, the denominator only, or both numerator and denominator?

12. If you multiply both numerator and denominator by the same number, what effect has it on the value of the fraction?

Find these products:

13. $7 \times \frac{2}{15}$ **15.** $38 \times \frac{5}{7}$ **17.** $2 \times \frac{5}{12}$ **19.** $83.4 \times .58$
14. $4 \times \frac{5}{9}$ **16.** $14 \times \frac{3}{11}$ **18.** $6 \times \frac{1}{9}$ **20.** $.749 \times .023$

Find the cost of each of the following:

21. 42 yds. at 38¢ a yd. **23.** 31 gals. at 76¢ a gal.
22. 162 lbs. at 89¢ a lb. **24.** 43 hrs. at 72¢ an hr.

25. By what number have you multiplied a number if you move the decimal point two places to the right, or if you add two 0's?

26. If you move the decimal point two places to the right and then divide the result by 4, by what have you multiplied the number?

27. Can you discover a short cut for multiplying by 25? By 50?

28. Find the value of: 584 × 50; 3612 × 25; 8248 × $12\frac{1}{2}$.

29. If you move the decimal point two places to the right and then divide by 3, by what have you multiplied the number?

30. Can you discover a short method of multiplying by $33\frac{1}{3}$? By $66\frac{2}{3}$? By $16\frac{2}{3}$?

31. Find 38169 × $33\frac{1}{3}$; 4578 × $66\frac{2}{3}$; 192114 × $16\frac{2}{3}$.

Find the per cent equal to:

32. 50% of 40%
33. 25% of 30%
34. 10% of 60%
35. 20% of 35%
36. 15% of 40%
37. 30% of 70%

Aliquot parts of 100%. Numbers that are contained in 100% a whole number of times are called aliquot parts of 100%. Some of them and their multiples are very important in business. Those used most often are:

Fifths	*Sixths*	*Eighths*
20% = $\frac{1}{5}$	$16\frac{2}{3}$% = $\frac{1}{6}$	$12\frac{1}{2}$% = $\frac{1}{8}$
40% = $\frac{2}{5}$	$33\frac{1}{3}$% = $\frac{1}{3}$	25% = $\frac{1}{4}$
60% = $\frac{3}{5}$	50% = $\frac{1}{2}$	$37\frac{1}{2}$% = $\frac{3}{8}$
80% = $\frac{4}{5}$	$66\frac{2}{3}$% = $\frac{2}{3}$	$62\frac{1}{2}$% = $\frac{5}{8}$
	$83\frac{1}{3}$% = $\frac{5}{6}$	75% = $\frac{3}{4}$
		$87\frac{1}{2}$% = $\frac{7}{8}$

Illustration. Find $16\frac{2}{3}$% of 384. Instead of a long multiplication, divide 384 by 6. The answer is 64.

Class Exercises

Find the value of:

1. $33\frac{1}{3}$% of 600
2. $37\frac{1}{2}$% of 800
3. 75% of 400
4. $66\frac{2}{3}$% of 900
5. $83\frac{1}{3}$% of 1200
6. $62\frac{1}{2}$% of 1600
7. $16\frac{2}{3}$% of 600
8. $87\frac{1}{2}$% of 2400
9. 50% of 1100

10. $12\frac{1}{2}\%$ of 600
11. 60% of 500
12. 20% of 700
13. 80% of 450
14. 40% of 600
15. 25% of 200

Optional Exercises

16. $12\frac{1}{2}\%$ of 384
17. $66\frac{2}{3}\%$ of 744
18. $62\frac{1}{2}\%$ of 552
19. $37\frac{1}{2}\%$ of 856
20. 40% of 380
21. $87\frac{1}{2}\%$ of 944
22. 60% of 585
23. $83\frac{1}{3}\%$ of 582
24. 50% of 762
25. $33\frac{1}{3}\%$ of 651
26. 80% of 475
27. 75% of 844

Honor Work

28. 40% of 389
29. $87\frac{1}{2}\%$ of 573
30. $12\frac{1}{2}\%$ of 341
31. 50% of 81.92
32. 25% of 7.163
33. $62\frac{1}{2}\%$ of 84.3
34. $33\frac{1}{3}\%$ of 61.7
35. $83\frac{1}{3}\%$ of .0052
36. $87\frac{1}{2}\%$ of 8.005

EXPONENTS IN MULTIPLICATION

Thought problem. You have already learned that a^2 means $a \cdot a$, that a^3 means $a \cdot a \cdot a$ and that a^4 means $a \cdot a \cdot a \cdot a$. See if you can discover for yourself what the answer would be if you multiplied a^3 by a^4. After you have found your answer, read the solution below.

How many a's are multiplied together to make a^3? To make a^4? If we write them along on a line, how many a's shall we have multiplied together? How can we write the answer? What did you do to the exponents 3 and 4 to get your answer?

$$a^3 \cdot a^4 = (a \cdot a \cdot a)(a \cdot a \cdot a \cdot a) = a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a = a^7$$

Quantities which when multiplied together give a certain product are called the *factors* of that product. a^3 and a^4 are factors of a^7.

HISTORICAL NOTE ON EXPONENTS: The present form of writing exponents was introduced by Descartes in 1637, but under other less convenient forms, many of the laws of exponents had been discovered at an earlier date. Oresme (1360), Chuquet (1484), and Stevin (1585) used symbols for exponents and even generalized them by the use of fractional and negative exponents.

Rule. **To find the exponent of each letter in a product, add all the exponents of that letter in the factors.**

$$a^m \cdot a^n = a^{m+n}$$

Class Exercises

Multiply:

1. $a^2 \cdot a^5$
2. $M^3 \cdot M^7$
3. $x^4 \cdot x^5$
4. $r^5 \cdot r^8$
5. $n^6 \cdot n^5$
6. $h^7 \cdot h$
7. $x \cdot x^5$
8. $x \cdot x$
9. $x^4 \cdot x^2$
10. $w \cdot w^3$
11. $B_1^2 \cdot B_1^3$
12. $k_1 \cdot k_1^2$
13. $y^3 \cdot y$
14. $A_1^3 \cdot A_1^4$
15. $p_2 \cdot p_2$
16. $6^2 \cdot 6^3$
17. $5^5 \cdot 5^4$
18. $2 \cdot 2^7$
19. $3^5 \cdot 3^8$
20. $1^3 \cdot 1$

Optional Exercises

21. $x^3 \cdot x \cdot x^5$
22. $m_1 \cdot m_1 \cdot m_1^5$
23. $n^4 \cdot n^2 \cdot n$
24. $x \cdot x^2 \cdot x^3$
25. $10 \cdot 10^2 \cdot 10^3$
26. $8^2 \cdot 8^5 \cdot 8^3$
27. $w_1^3 \cdot w_1 \cdot w_1^2$
28. $m_2^2 \cdot m_2^2 \cdot m_2^2$
29. $r_1^2 \cdot r_1 \cdot r_1^4$

Honor Work

30. $a^x \cdot a^y$
31. $m^a \cdot m^b$
32. $x^a \cdot x^2$
33. $y_1 \cdot y_1^m$
34. $x^{a+1} \cdot x^{a-1}$
35. $x^n \cdot x \cdot x^3$

Investigating the multiplication of signed numbers.

1. The temperature is rising 3° an hr. (+ 3°) and is now 0°. If it continues at that rate, what will the temperature be in 4 hrs. (+ 4 hrs.)? Evidently it will be 12° above 0 (+ 12°). $[(+3)(+4) = +12]$

2. The temperature is falling 3° an hr. (− 3°) and is now 0°. What was the temperature 4 hrs. ago (− 4 hrs.) if it has continued falling at the same rate? Evidently it was 12° above 0 (+ 12°). $[(-3)(-4) = +12]$

3. The temperature is rising 3° an hr. (+ 3°) and is now 0°. If it has been rising at that rate, what was the temperature 4 hrs. ago (− 4 hrs.)? Evidently it was 12° below zero (− 12°). $[(+3)(-4) = -12]$

4. The temperature is falling 3° an hr. (− 3°) and is now 0°. If it continues at that rate, what will the tempera-

ture be in 4 hrs. ($+4$ hrs.)? Evidently it will be 12° below zero ($-12°$). $[(-3)(+4) = -12]$

What sign has the product when both signs are plus? When both signs are minus? When one is plus and one minus?

What sign has the product when both signs are alike? When they are unlike?

Rule. To multiply two signed numbers:

1. Multiply the numbers without considering the sign.

2. If the signs are alike, make the product plus; if the signs are unlike, make the product minus, or,

In multiplication, like signs give plus; unlike signs give minus.

Class Exercises

1. $(+3)(+7)$
2. $(-5)(+2)$
3. $(-4)(-3)$
4. $(+6)(-8)$
5. $-4(+6)$
6. $-3(-8)$
7. $+2(-3)$
8. $3(-7)$
9. $+(-3)$
10. $-1(+9)$
11. $-(-4)$
12. $-1(1)$
13. $(+3)(+0)$
14. $-2(0)$
15. $-0(+7)$
16. $(0)(0)$

Multiply:

17. $\begin{array}{r} -5 \\ \underline{+3} \end{array}$
18. $\begin{array}{r} +6 \\ \underline{3} \end{array}$
19. $\begin{array}{r} -8 \\ \underline{-7} \end{array}$
20. $\begin{array}{r} 5 \\ \underline{-4} \end{array}$
21. $\begin{array}{r} +6 \\ \underline{-2} \end{array}$
22. $\begin{array}{r} -9 \\ \underline{-4} \end{array}$
23. $\begin{array}{r} +4 \\ \underline{0} \end{array}$
24. $\begin{array}{r} +0 \\ \underline{-3} \end{array}$

25. $(3a)(4)$
26. $(4w)(6w)$
27. $(a^2)(a^3)$
28. $(4M^5)(5M)$
29. $(5x^3y^7)(2xy)$
30. $(5x)(-7x)$
31. $(-x^3)(+x^4)$
32. $(4p^3)(-5p^4)$
33. $(x_1x_2)(-x_1x_2)$
34. $(-3kn)(0)$

35. $3a^2b^3(5ab^2)$
36. $-2mn(+5m^3n)$
37. $-8w_1w_2(-3w_1{}^3w_2{}^2)$
38. $3h(-5h)(0)$
39. $-4x^4(-3x^3)$

Optional Exercises

40. $(-2x)(-2x)(-2x)$
41. $(-5a^2b)(+2ab^3)(-2a)$
42. $(7x)(2y)(z)$
43. $(-3x^2yz)(-9x^3yz^9)(0)$
44. $(-4x^7yz^8)(2xz^3)(-4y)$
45. $-5^4(5^7)$
46. $-3^5(-3^5)$
47. $(-3^5)^2$
48. $(-3^5)^3$
49. $(.3)(-2)$
50. $(-1)^2$
51. $(-1)^3$
52. $(-1)^4$
53. $(-1)^{101}$
54. $(.3a^2)^3$

Honor Work

55. $a^p \cdot a^q$
56. $a^n \cdot a^3$
57. $a^{n-1}(-a)^4$
58. $-a^n \cdot a$
59. $-a^{n-1}(-a)$
60. $-2^2(-2)$
61. $-2^x \cdot 2^y$
62. $+3^r(-3^r)$
63. $(-10)^5$
64. $(-.3a)(+.02a^2)(-.1a)$
65. $5w_1w_2(-2w_1w_3)(-w_2w_3)$
66. $(x^n)^2$

67. What is the value of $(-1)^x$ when x is an odd number? When x is an even number?

68. If x is decreasing, will x^2 also decrease when x is (*a*) positive? (*b*) Negative? (*c*) When x changes from positive to negative?

Exponents in Astronomy

The distance to one of the nearest stars is about 26,000,000,000,000 mi., and to another star about 5,200,000,000,000,000 mi. How many times as far away is the second star as the first?

THE NEBULA OF ANDROMEDA

Nebulas are other universes far beyond the stars of our universe. They are so far away that the stars in them appear as simply a cloud. Andromeda is about 900,000 light years away or about 5 × 10^{18} miles from us. Write the number in full.

Astronomers find it much easier to handle such extremely large numbers when they are expressed as powers of 10. The power of 10 used equals the number of 0's for:

Power of Ten	*Exponent*	*Number of Zeros*
$10^1 = 10$	1	1
$10^2 = 100$	2	2
$10^3 = 1000$	3	3
$10^4 = 10000$	4	4

The first distance then is 26×10^{12}, the second 52×10^{14}. In this form it is easy to see that the second number is larger, and to determine how many times as large. Find the answer.

Scientists often write very large or very small numbers with the decimal point after the first figure. In this case we have:

$$\begin{aligned} 260 &= 2 \text{ hundred and } 60 = 2.6 \times 100 = 2.6 \times 10^2 \\ 2600 &= 2 \text{ thousand and } 600 = 2.6 \times 1000 = 2.6 \times 10^3 \\ 26{,}000{,}000{,}000{,}000 &= 2.6 \times 10{,}000{,}000{,}000{,}000 \\ &= 2.6 \times 10^{13} \\ 5{,}200{,}000{,}000{,}000{,}000 &= 5.2 \times 1{,}000{,}000{,}000{,}000{,}000 \\ &= 5.2 \times 10^{15} \end{aligned}$$

1. A light year, the distance light travels in a year, is about 6×10^{12} mi. Express these distances in miles, using powers of 10:

(*a*) 100 light years (*c*) 1,000,000 light years

(*b*) 80,000 light years (*d*) 20,000,000 light years

2. If there are about 10,000 light waves to 1 in., approximately how many light waves are on their way here from a star 1,000,000 light years from us?

Polynomials

What is a term? What signs tell you where one term ends and the next begins? How many terms are in the expression $5x$? In $7x^2y^3z$? In $x + 1$? In $5a - 2b + 3$?

An expression having just one term is called a ***monomial.***

An expression having more than one term is called a ***polynomial.*** When the polynomial has just two terms, it is a ***binomial;*** when it has three terms, a ***trinomial.*** You will see that polynomials have some resemblance to numbers containing more than one figure. Here we have three *terms* whose sum is 13. How many of them must you multiply by 2 in order to make the sum twice as great, or 26? Here we have three *factors* whose product is 30. How many of them must you multiply by 2 to make the product twice as great, or 60? If you multiply every factor by 2 and write 4 × 6 × 10, by what number have you multiplied the product?

$$\begin{array}{l} 3+4+6=13 \\ ?+?+?=26 \end{array}$$

$$\begin{array}{l} 2\times 3\times 5=30 \\ ?\times ?\times ?=60 \end{array}$$

An expression is multiplied by a number when all of its terms are multiplied by that number or when one of its factors is multiplied by the number.

How to multiply a polynomial by a monomial. Do you multiply every figure of the 342 by the 2? Why do you begin at the right instead of at the left? Begin at the left, and see if you get the same answer. Now multiply 342 by 7. Can you begin at the left in this example? Why not?

$$\text{Multiply: } \begin{array}{r} 3\ \ 4\ \ 2 \\ 2 \\ \hline \end{array}$$

When we multiply in algebra, we have no number to carry over from one term to the next, so we can, and by custom we do, begin at the left. Otherwise the multiplication is done just as in arithmetic.

$$\begin{array}{r} 3\ \ 4\ \ 2 \\ 2 \\ \hline 6\ \ 8\ \ 4 \end{array} \qquad \begin{array}{r} 3a+4b+2c \\ 2 \\ \hline 6a+8b+4c \end{array}$$

Beginning at the left, we multiply 3 a by 2, then 4 b by 2, and finally 2 c by 2.

In the same way: $-3a^2b(2a^3-4ab^2+5ab^3) = -6a^5b+12a^3b^3-15a^3b^4$.

Check by substituting any convenient numbers for the letters.

Illustration. Multiply $3x^2-2x-5$ by $-3x$ and check by substituting 2 for x. Note: $x=1$ will check signs and coefficients, but will not check exponents, for $x^2=1$, $x^3=1$

$x^4 = 1$. So a wrong exponent would give the same value as the right one.

$$\begin{array}{rcrr}
3x^2 - 2x \qquad - 5 & = & 3 \cdot 4 - 2 \cdot 2 - 5 = & +3 \\
-3x & = & -3 \cdot 2 = & \underline{-6} \\
\hline
-9x^3 + 6x^2 + 15x & = & -9 \cdot 8 + 6 \cdot 4 + 15 \cdot 2 & -18 \leftarrow \\
 & = & -72 + 24 + 30 = & -18 \leftarrow
\end{array}$$

Class Exercises

Multiply and check:

1. $4(2x - 7)$
2. $-5(3a - 4b)$
3. $-8x(5x + 4x^3)$
4. $4k(3y - 2k)$
5. $-PQ(P^2 - Q^2)$
6. $2A^2x^3(-Bx + Ax^2)$
7. $-cz(-z - 2z^3)$
8. $-5M^2N(3M + 2N^2)$
9. $2xyz(x^2z - 3yz^3)$
10. $-2c^2d(c^3 + d^5)$
11. $-4ab(2a^3 - 4ab^4 + 3a^2b)$
12. $2n^2(-3an + 6bn^2 - 2c)$
13. $-2xy(4x^2 + 5xy - 2y^2)$
14. $3a^2b^3(5a^4b - 2ab^5 - 6a^3b^2)$
15. $-4y^2(-5y^3 + 3y^2 - y)$
16. $-5x^3y^3(3ax^2y - axy^4 + b)$
17. $2a^2b^3c^5d(3ab^3cd^2 - cx) - 3a$
18. $3a^3 - 5a^2b(2a^2 - 3ab^4)$

Optional Exercises

19. $3a - 1(2a - 3) - (a + 4)$
20. $k_1(k_2 + k_1{}^2)$
21. $-w_1w_2(w_1{}^2 - 5w_2{}^3)$
22. $2\pi r(r_1 + r_2)$
23. $(2x + y)3y - 4(y - 2x)$
24. $-3x^4(4m + 0 + x^3)$
25. $2\pi r_1r_2(r_1{}^2 + r_2{}^2)$
26. $(0)(3x^2y - 2y^2z) + 4xyz$
27. $\frac{1}{2}x^2y^3(\frac{1}{3}x^3 + \frac{1}{2}y - \frac{1}{2})$
28. $-4x(-2x + 3y)(-5y)$

Honor Work

29. $-.3x^2(2.1x + 1.2)$
30. $3ab^2(1\frac{2}{3}ab - \frac{1}{3}b^3)$
31. $-1\frac{3}{4}M_1(\frac{4}{7}M_1 + \frac{1}{2}M_2)$
32. $c^x(2c^{2x} - 3c^x + 1)$
33. $y(ay^b - 2y^{b-1} + 3y)$
34. $-x^{n-1}(x^2 + x^{n+1} - 3x^n)$

Thought Questions

In multiplying a monomial by a binomial, which should be enclosed in a parenthesis? Why?

In what way is a polynomial changed when it is multiplied by -1?

MULTIPLICATION IN EQUATIONS

Illustration. Solve and check: $5(x - 2) = 4(x + 1)$

$5(x - 2) = 4(x + 1)$	The given equation.
$5x - 10 = 4x + 4$	Removing parentheses.
$4x - 10 = 4x - 10$	Any quantity equals itself.
$x = 14$	If equals are subtracted, etc.

Check by substituting the answer in the *original* equation.

$5(14 - 2)$	$4(14 + 1)$	Substitution.
5×12	4×15	

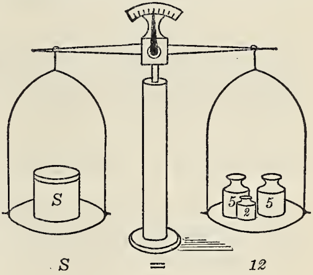
$$60 = 60$$

Exercises

Solve and check:

1. $2(x + 4) - x = 11$
2. $4(y - 3) = 3y$
3. $2(x - 4) - x = 1$
4. $3(r + 2) = 2(r + 5)$
5. $2(3v - 1) = 5(v - 2)$
6. $3(2 + 3x) - 4(2x - 1) = 1$
7. $5t - 2(1 + 2t) = 0$
8. $3 - 2(3 - 4t) = 7(t + 2)$
9. $k - 3(1 + 2k) = 2(2 - 3k)$
10. $(M + 1)M - 3 - M^2 = 3$

Keeping the balance. Remember that the equation is a balance. Both sides must always be equal. In the figure the weights on the right pan balance the can of sugar on the left pan. What change must we make in the weights if we double the amount of sugar in the left pan? If we multiply the amount of sugar by 3? By 4? By 10?



Axiom 3. *To keep the balance both members of an equation must be multiplied by the same number, or:*

If equals are multiplied by equals, the results are equal.

(=s × =s are =.)

How to solve fractional equations. If you multiply one side of an equation by 2, what must you do to the other side? If $\frac{x}{3}$ were on one side of an equation, what must you multiply both sides by to get rid of the denominator? What must you multiply by to get rid of the denominator in $\frac{3x}{5}$? In $\frac{x+1}{6}$?

Illustration 1. Solve and check: $2x = \frac{3x}{2} + \frac{5}{2}$

We can get rid of the denominators if we multiply both sides of this equation by 2.

	$2x = \frac{3x}{2} + \frac{5}{2}$	The original equation.
Multiply by 2:	$4x = 3x + 5$	If equals are multiplied, etc.
	$\underline{3x = 3x}$	Any quantity equals itself.
	$x = 5$	If equals are subtracted, etc.

Check:

$$\begin{array}{r|l} 2 \cdot 5 & \frac{15}{2} + \frac{5}{2} \\ 10 & \frac{20}{2} \\ \multicolumn{2}{c}{10 = 10} \end{array}$$

Substitution.

NOTE: In checking, do not multiply by 2. It is always better to simplify the check in as different a way as possible from that used in solving the equation. Otherwise, you are likely to make the same mistake in both, and a wrong root seems to check.

Illustration 2. Solve: $\frac{2x}{3} + \frac{x}{5} = 26$

To get rid of the first denominator we must multiply by 3 and to get rid of the second by 5. So to save time, we multiply by 15 and get rid of both at once.

Multiplying by 15: $15\left(\frac{2x}{3}\right) + 15\left(\frac{x}{5}\right) = 15(26)$

$$\begin{aligned} 10x + 3x &= 390 \\ 13x &= 390 \\ x &= 30 \end{aligned}$$

Thought Questions

What is the smallest multiplier that will get rid of all three denominators in $\frac{x}{8} + \frac{x}{6} - \frac{x}{4} = 2$? Is it necessary to multiply by the product of 8, 6, and 4, or will a smaller number do? The smallest number that will get rid of all denominators is called the lowest common denominator (L.C.D.) of the equation.

Class Exercises

Solve and check:

1. $\frac{x}{5} = 3$
2. $\frac{x}{3} = 8$
3. $\frac{x+4}{5} = 2$
4. $\frac{a-3}{5} = 1$
5. $t - \frac{4t}{5} = 3$
6. $2x - \frac{5x}{3} = \frac{4}{3}$
7. $\frac{r}{2} - \frac{r}{3} = 20$
8. $\frac{y}{4} - \frac{y}{8} = 3$
9. $3w = \frac{5w}{2} - \frac{1}{2}$
10. $\frac{x}{2} - \frac{x}{3} = 7$
11. $\frac{3x}{4} - \frac{2x}{3} = \frac{3}{4}$
12. $\frac{3x}{4} - \frac{2x}{3} = 5\frac{1}{2}$
13. $\frac{m}{2} - 5 = \frac{3m}{7}$
14. $\frac{w-1}{2} = \frac{3w}{7}$
15. $\frac{5s}{6} - \frac{1}{2} = \frac{2s}{3}$
16. $x - \frac{1}{3} = \frac{2x}{3} - 2$
17. $4x - \frac{7x}{5} = 3 + \frac{12x}{5}$
18. $\frac{x}{4} + \frac{x}{3} - \frac{x}{2} = 3$

Optional Exercises

19. $.7x = 1.4$
20. $.3m = 1.8$
21. $k + .3 = 2.5$
22. $.03r + 5 = 20$
23. $v + .8v = .72$
24. $.05 - x = .4 - 1.07x$
25. $.05(x + .2) = .96$
26. $.001(x + .1) = .35$
27. $\frac{s-2}{3} - \frac{s+1}{4} = 4$
28. $\frac{x-5}{4} + 1 = \frac{x-3}{5}$
29. $\frac{x+3}{2} + \frac{4-x}{3} - 1 = 0$
30. $\frac{x+4}{3} - \frac{x-2}{4} = \frac{5}{6}$

Honor Work

31. $\frac{x+2}{x-3} = \frac{7}{4}$

32. $\frac{x-5}{x+5} + 8 = -1$

33. $\frac{7}{x} - \frac{8}{2x} = 3$

34. $\frac{H-2}{H} - 1 = \frac{1-2H}{2H}$

35. $\frac{8}{x-2} - \frac{3}{2x-4} = \frac{13}{2}$

36. $\frac{5}{x-3} + \frac{2}{3-x} = 3$

THE MONOMIAL FACTOR

Henry Adams found the distance around 2 pulleys whose diameters were 7.34 in. and 2.66 in. Here is Henry's work:

$$3.14 \times 7.34 + 3.14 \times 2.66 = 23.0476 + 8.3524 = 31.4$$

Fred Beale pointed out that he would have gotten the same result if he had simply multiplied 3.14 by 10. Can you discover why this is so?

Very often it is necessary to multiply several numbers by the same number. The walls of a basement may all be the same thickness, so to find the amount of cement needed, the areas of all of these walls are multiplied by the same number. Also a commission merchant must multiply all sales by the same fraction to get his commission. We shall now learn the short cut for this multiplication.

Exercises

Multiply:

1. $3x(a+b)$

2. $-2ab^2(3a-2b)$

3. $3x^2(2x^2-x-4)$

4. $\frac{1}{2}h(a+b)$

5. $p(1+rt)$

6. $5a^2b(2a^3-3a^2b+ab^2)$

7. $-4xy^2z(3x^2+2xy^2-y^2z^3)$

8. $3xy(-7x^2y^2+xy-2y^3)$

9. $2\pi r(r+h)$

10. $\frac{1}{6}h(a+b+4m)$

Multiply $3a^2x$ by $2x-y+4$. The result is $6a^2x^2 - 3a^2xy + 12a^2x$. Now if you were given the answer only, could you discover what quantities were multiplied together to give it? Look first at the monomial factor $3\,a^2x$. What is there about the answer that would suggest the 3? The a^2? The x? If you knew that $3\,a^2x$ was multiplied by some other quantity to give the product $6\,a^2x^2 - 3\,a^2xy + 12\,a^2x$, how could you find that other?

Illustration. Factor: $9\,x^3 - 6\,x^2 + 15\,x^4$

The largest number that will go into 9, -6, and 15 is 3.

The largest power of x that will go into x^3, x^2, and x^4 is x^2.

The monomial factor then is $3\,x^2$.

Dividing $3\,x^2$ into $9\,x^3 - 6\,x^2 + 15\,x^4$, the other factor is $3\,x - 2 + 5\,x^2$. So $9\,x^3 - 6\,x^2 + 15\,x^4 = 3\,x^2(3\,x - 2 + 5\,x^2)$.

Class Exercises

Find the factors of:

1. $3\,a + 3\,b$
2. $2\,x - 6\,y$
3. $ax - ay$
4. $3\,x - 6$
5. $6\,y + 3$
6. $a^2b - ab$
7. $xy + yz$
8. $abc + ab$
9. $7\,x^2 - 14\,x$
10. $2\,ax - 2\,ay$
11. $3\,mr + 6\,ms$
12. $\frac{1}{2}\,an + \frac{1}{2}\,np$
13. $2\,\pi r^2 + 2\,\pi rh$
14. $\pi r^2 + \pi rs$
15. $6\,a^3 - 3\,a^2$
16. $vt + \frac{1}{2}\,at^2$
17. $x^2y + xy^2$
18. $2\,x + 4\,x^2$
19. $15\,y^3 - 5\,y^2$
20. $\frac{1}{2}\,hb_1 + \frac{1}{2}\,hb_2$
21. $-3\,x - 9\,y$
22. $5\,x + 5\,y - 15\,z$
23. $ax - ay + a$
24. $3\,x^3 - 6\,x^2 + 6\,x$
25. $\frac{1}{6}\,hB + \frac{1}{6}\,hb + \frac{4}{6}\,hm$
26. $\frac{1}{4}\,a + \frac{1}{4}\,b + \frac{1}{4}\,c + \frac{1}{4}\,d$
27. $15\,a^3b^4 - 20\,a^2b^5 + 5\,ab^3$
28. $x^{16} + x^{11} + x^9$
29. $\frac{1}{2}\,\pi R^2H + \frac{1}{2}\,\pi r^2H + \frac{1}{2}\,\pi RrH$
30. $Pr + Pr^2 + Pr^3$
31. $\frac{1}{3}\,hB + \frac{1}{3}\,hb + \frac{1}{3}\,h$

Thought question. How can you check your factors?

Optional Exercises

32. $.3\,x + .6\,y + 1.2$
33. $1.2\,x^3 - .9\,x^4 + 1.5\,x^5$
34. $12\,x^2y + 15\,xyz - 10\,yz^2$
35. $\frac{1}{2}\,ab + \frac{1}{3}\,bc + \frac{3}{4}\,abc$
36. $8\,a^2b^3 - 12\,a^5b^3 + 4\,a^2b^5$
37. $(a + b)x + (a + b)y$
38. $(m - n)x + (m - n)y$
39. $(a - b)x - (a - b)y$
40. $a(p - q) + (p - q)$
41. $a(b - c) - (b - c)$

Honor Work

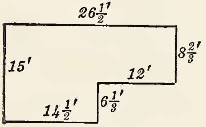
42. $ax + bx + ay + by$
43. $mn - mp + kn - kp$
44. $x^3 + x^2 + x + 1$
45. $mx - nx - my + ny$
46. $ab - ac - b + c$
47. $x^{n+2} - x^{n+1} + x^n$
48. $y^{2a} + y^a$
49. $2^{17} + 2^{15}$
50. $(x + 3)^3 - (x + 3)^2$
51. $x^{n+3} + x^{n+2} + x^{n+1} + x^n$

Factoring in Business

Find the results with the least possible work:

1. $3.1416 \times 5.734 + 3.1416 \times 4.266$
2. $647 \times 12.893 - 647 \times 2.893$
3. $13.74 \times 84.31 + 13.74 \times 55.69$
4. $62\frac{1}{2}\% \times 347 + 62\frac{1}{2}\% \times 453$
5. $87\frac{1}{2}\% \times 54.91 + 87\frac{1}{2}\% \times 73.73$
6. $33\frac{1}{3}\% \times 743 - 33\frac{1}{3}\% \times 152$

7. A commission merchant receives a commission of $3\frac{1}{2}\%$ on each of these sales: 200 lbs. coffee at 40¢ a lb., 100 doz. eggs at 52¢ a doz., and 600 bu. potatoes at \$1 a bu. Find his total commission.



8. A house is to be clapboarded all around to a height of $9\frac{1}{3}$ ft. If the dimensions of the ground plan are as shown in the diagram, find the total area to be clapboarded.

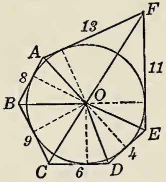
9. A cellar in the form of a rectangle, $28\frac{1}{2}$ ft. by $13\frac{1}{2}$ ft., is to have walls $8\frac{1}{2}$ ft. high and 1 ft. thick. How many cubic feet of material will be needed?

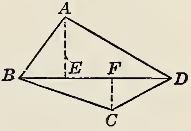
10. Paul Brown receives 6% interest on 3 mortgages for \$2643, \$1574, and \$4383. What yearly income does he receive from them?

11. How many square feet of floor will be covered by boards all 5 in. wide and of lengths of 11 ft. 4 in., 8 ft. 6 in., 12 ft. 10 in., 11 ft. 6 in., and 9 ft. 10 in.?

12. A circular running track has an external diameter of 1223 ft. and an internal diameter of 1213 ft. Using $\pi = 3.1416$, find how much farther the boy who travels around the outer edge must run than the boy who travels around the inner edge. The formula is $\pi D - \pi d$.

13. A circle of radius 6 just touches all of the sides of polygon $ABCDEF$. If $AB = 8$, $BC = 9$, $CD = 6$, $DE = 4$, $EF = 11$, and $FA = 13$, find the area of the polygon. The radius is the altitude of each triangle. How would you find the area of triangle AOB? Of triangle BOC? Of triangle COD?





14. Find the area of $ABCD$ by adding the areas of the two triangles, if $BD = 13.82$ in., altitude $AE = 5.74$ in., and altitude $CF = 4.26$ in.

How to multiply binomials.

Illustration. Multiply: $3x + 4$ by $2x - 1$ Check.

$$\begin{array}{rr}
3x + 4 & = 10 \\
\underline{2x - 1 } & = \underline{3} \\
6x^2 + 8x & 30 \\
\underline{-3x - 4} & \uparrow \\
6x^2 + 5x - 4 & \downarrow \\
24 + 10 - 4 & = 30
\end{array}$$

The multiplication is done as in arithmetic except that we begin at the left. $2x$ times $3x$ is $6x^2$, $2x$ times 4 is $8x$, -1 times $3x$ is $-3x$, and -1 times 4 is -4. Then add. Check by substituting 2 for x.

Exercises

Multiply:

1. $\begin{array}{r} 2a + 3 \\ \underline{3a + 4} \end{array}$ **2.** $\begin{array}{r} 3m - 5 \\ \underline{2m + 6} \end{array}$ **3.** $\begin{array}{r} 5r - 3 \\ \underline{2r - 7} \end{array}$ **4.** $\begin{array}{r} k - 4 \\ \underline{2k + 1} \end{array}$

5. $(x + 5)(2x - 3)$
6. $(3v - 7)(2v + 9)$
7. $(5L + 2)(4L - 3)$
8. $(2a + 3b)(4a - b)$
9. $(3c - b)(2c - 5b)$
10. $(x^2 + 7)(2x^2 - 5)$
11. $(H + 4J)(H - 4J)$
12. $(5x^3 - 2y^2)(3x^3 - 4y^2)$
13. $(x + .04)(x - .03)$
14. $(.5N - 3R)(4N + .1R)$
15. $(ax + by)(2ax - 3by)$
16. $(3x^2y - 5xy^3)^2$
17. $(5x - 2y)^2$
18. $(\frac{1}{2}a + \frac{1}{3}b)^2$
19. $(.2x - .01)^2$
20. $(2.5m + 1.2k)^2$

NOTE. For the multiplication of polynomials, see page 362.

DIVISION

Review of Arithmetic

Divide, and check by multiplication:

1. 4384 ÷ 16
2. 9576 ÷ 38
3. 2735 ÷ 63
4. 48094 ÷ 346
5. 28968 ÷ 852
6. 72102 ÷ 591
7. 100968 ÷ 4207
8. 391265 ÷ 7017
9. 908401 ÷ 4456

10. How do you divide a number by 10? By 100? By 1000?

11. Find the value of 68200 ÷ 100; 45321 ÷ 1000.

12. If you point off three places in a number, by what have you divided it?

13. By what have you divided a number, if you point off two places?

14. If you point off two places and then multiply the result by 2, by what have you divided the number?

15. Can you discover an easy way of dividing by 50? By 25?

16. Find 831,000 ÷ 50; 694 ÷ 25.

17. If you point off two places and multiply the result by 3, by what have you divided the number?

18. Can you discover an easy way of dividing by $33\frac{1}{3}$? By $16\frac{2}{3}$?

19. Find 384,100 ÷ $33\frac{1}{3}$; 71962 ÷ $33\frac{1}{3}$; 48211 ÷ $16\frac{2}{3}$.

Divide:

20. .344 ÷ 8
21. 76.38 ÷ 19
22. 462 ÷ .007
23. 552 ÷ .06
24. 1.391 ÷ 1.3
25. 83.25 ÷ .015

What per cent of:

26. 42 is 21
27. 624 is 78
28. 24480 is 153
29. 5750 is 2185
30. 1 is .04
31. .07 is .14

ALGEBRAIC DIVISION.

Division is the opposite of multiplication. By what number do we multiply 3 to get 15? Then 15 ÷ 3 = what?

In division we find one of two numbers when their product and the other number are known. We sometimes use the familiar sign ÷, but in algebra we more often indicate division by the fractional form. For example $\frac{14}{5}$ means 14 divided by 5.

Just as in arithmetic, the number you divide is called the ***dividend;*** the number you divide by, the ***divisor;*** and the answer, the ***quotient.***

Investigating division. By what number would you multiply:

1. $+3$ to get $+15$? Then $+15 \div +3 = ?$
2. -3 to get -15? Then $-15 \div -3 = ?$
3. $+3$ to get -15? Then $-15 \div +3 = ?$
4. -3 to get $+15$? Then $+15 \div -3 = ?$

What sign has the answer when both signs are plus? When both are minus? When both signs are alike?

What sign has the answer when one sign is plus and the other is minus? When the signs are unlike?

Rule. If two numbers have like signs, their quotient is positive; if they have unlike signs, their quotient is negative.

Exercises

1. $+6 \div +2$
2. $+8 \div -2$
3. $-10 \div -5$
4. $-8 \div +4$
5. $-12 \div -6$
6. $+9 \div -3$
7. $-5 \div -5$
8. $+7 \div -7$
9. $+6 \div -2$
10. $-18 \div +6$
11. $-16 \div +2$
12. $+0 \div -3$
13. $\dfrac{-11}{-1}$
14. $\dfrac{-9}{+1}$
15. $\dfrac{+6}{-6}$
16. $\dfrac{+0}{+7}$
17. $\dfrac{-5}{+4}$
18. $\dfrac{+3}{-5}$

Exponents in division. In multiplication, what do you do with the exponents of a letter? What is the value of $a^5 \cdot a^7$?

By what would you multiply a^3 to get a^7? Then $a^7 \div a^3 =$ what?

If you add the exponents of a letter in multiplication, what would you expect to do with them in division, the opposite of multiplication?

$$\frac{a^7}{a^3} = \frac{a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a}{a \cdot a \cdot a} = a \cdot a \cdot a \cdot a = a^4$$

What did you do to the 7 and 3 to get 4?

Rule. The exponent of any letter in the quotient equals its exponent in the dividend minus its exponent in the divisor.

$$\frac{a^m}{a^n} = a^{m-n}$$

Illustration. Divide $-12\,a^3b^2c^5$ by $+3\,a^2bc^3$

$-12 \div +3 = -4,\ a^3 \div a^2 = a,\ b^2 \div b = b,$ and $c^5 \div c^3 = c^2$

$$-12\,a^3b^2c^5 \div +3\,a^2bc^3 = -4\,abc^2$$

Class Exercises

Divide:

1. x^5 by x^2
2. y^8 by y^4
3. a^9 by a^3
4. b^7 by b^5
5. c^4 by c
6. m^9 by m^4
7. $a^{13} \div a^5$
8. $L^4 \div L$
9. $x^5 \div x^5$
10. $n^{15} \div n^5$
11. $R^3 \div R$
12. $p^{10} \div p^2$
13. $\dfrac{4\,a_1{}^4}{2\,a_1{}^2}$
14. $\dfrac{6\,b^6}{2\,b^2}$
15. $\dfrac{9\,c^3}{3\,c}$
16. $\dfrac{8\,x^8}{4\,x^4}$
17. $\dfrac{10\,y^5}{2\,y}$
18. $\dfrac{12\,z^{12}}{6\,z^6}$

Optional Exercises

19. $(-3\,x^2y^3) \div (-xy)$
20. $(-12\,a^3b^4) \div (+3\,ab^3)$
21. $(+14\,c^8d^5) \div (-2\,c^5d^2)$
22. $(+18\,m^5n^9) \div (+6\,m^2n^4)$
23. $(-12\,x^{12}y^{10}) \div (4\,x^6y^5)$
24. $(+16\,R^{16}r^{16}) \div (+4\,R^4r^8)$
25. $-8\,x_1{}^3x_2{}^6x_3{}^9 \div (-2\,x_1{}^3x_2{}^4x_3{}^5)$
26. $+12\,a^7b^5c \div (-6\,a^7c)$
27. $-7\,\pi r^3h \div (-rh)$
28. $-9\,x^9y^9z^9 \div 5\,x^5y^9z^8$
29. $+2\,c^7x^5y^{10} \div 8\,c^2x^2y^5$
30. $-3\,A_1{}^7A_2{}^8A_3{}^{11} \div (-3\,A_1{}^7A_2{}^8A_3{}^{11})$

Honor Work

31. $\dfrac{x^{2n}}{x^n}$
32. $\dfrac{y^{n+5}}{y^4}$
33. $\dfrac{x^{2a}}{x^2}$
34. $\dfrac{a^{x+1}}{a^{x-1}}$
35. $\dfrac{2^{2n}}{2^n}$
36. $\dfrac{2^x}{2}$
37. $\dfrac{ax^a}{2\,x^2}$
38. $\dfrac{x^{6b}}{x^b}$
39. $\dfrac{x^{2c}}{x^3}$
40. $\dfrac{-3\,ay^{3a}}{-ay^a}$
41. $\dfrac{+6\,bz^7}{-2\,z^n}$
42. $\dfrac{ax^a}{bx^b}$

How to divide a polynomial by a monomial. To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

Illustration. Divide: $\dfrac{6x^3 - 3x^2 + 9x^5}{3x^2}$

$6x^3 \div 3x^2 = 2x$, $-3x^2 \div 3x^2 = -1$, and $9x^5 \div 3x^2 = 3x^3$

The answer is $2x - 1 + 3x^3$.

Exercises

Divide:

1. $\dfrac{3x + 6}{3}$
2. $\dfrac{5a - 15}{5}$
3. $\dfrac{-8t + 4}{4}$
4. $\dfrac{6y - 12}{-3}$
5. $\dfrac{3x^2 - 5x}{x}$
6. $\dfrac{5x^2 + 5x}{5x}$
7. $\dfrac{9x^2 - 15y^2}{-3}$
8. $\dfrac{-5a^2b + 10ab^3}{5ab}$
9. $\dfrac{2\pi r^2 + 2\pi rh}{\pi r}$
10. $\dfrac{-a + b - c}{-1}$
11. $\dfrac{2wh + 2mh}{2h}$
12. $\dfrac{-14x^3y^2 - 21x^2y^4}{-7x^2y^2}$
13. $\dfrac{-5an + 10bn}{-5cn}$
14. $\dfrac{\frac{1}{2}x^2 - \frac{1}{4}x}{\frac{1}{2}x}$
15. $\dfrac{15a^2bx - 10ab^2y}{-5ab}$

Thought Questions

How is a polynomial changed when you divide it by -1? Is the result the same as if you multiplied it by -1?

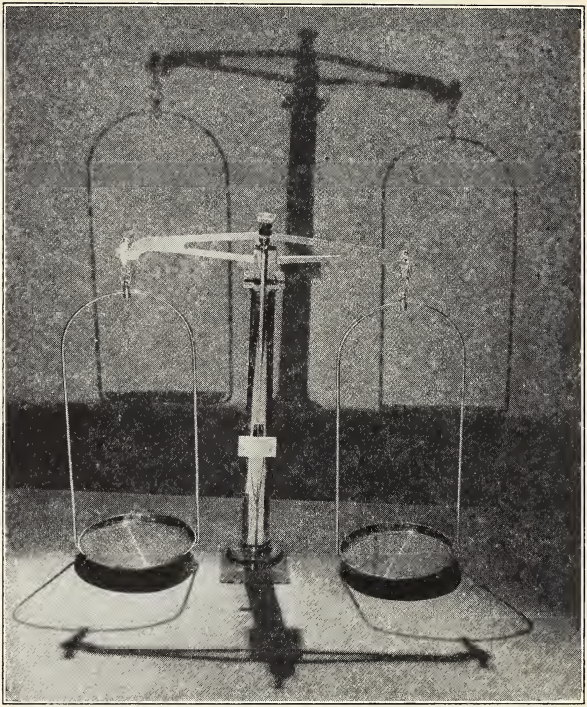
The balance once more. What must you do to the weight in the right pan if you divide the amount of sugar in the left pan by 2? If you divide it by 3? By 5?

Axiom 4. *To keep the balance, both members of an equation must be divided by the same number, or:*

If equals are divided by equals, the results are equal.

(=s ÷ =s are =.)

NOTE: For the division of polynomials, see page 364.



By Ewing Galloway.

THE BALANCE

DIVISION IN EQUATIONS

Illustration. Solve $5x - 7 = 2x + 5$.

$5x - 7 = 2x + 5$	The original equation.
$2x - 7 = 2x - 7$	Any quantity equals itself.
$3x = 12$	If equals are subtracted, etc.
$\frac{3x}{3} = \frac{12}{3}$	If equals are divided by equals, the results are equal.
$x = 4$	

Check:
$$\begin{array}{r|l} 5 \cdot 4 - 7 & 2 \cdot 4 + 5 \\ 20 - 7 & 8 + 5 \\ 13 = & 13 \end{array}$$
Substitution.

Class Exercises

Solve, giving axioms, and check:

1. $2x = 10$
2. $5x = 35$
3. $3x = 11$
4. $4x = -8$
5. $-7x = -21$
6. $3m + m = 20$
7. $5r - 2r = 18$
8. $2k - 7k = 15$
9. $3x + 4x = 10$
10. $4n - 7n = 6$
11. $5A + 8 = 13$
12. $9w = 12 + 5w$
13. $4y + 12 = 8$
14. $4 - 7t = 11$
15. $s - 15 = 6s$

16. $5x + 7 = 3x + 15$
17. $4 + 3h = 9 - 2h$
18. $3z + 10 = z + 10$
19. $4R - 7 = 13 - R$
20. $3v + 2 = 18 + 5v$
21. $2 - 4x = 12 + x$
22. $7 - r = 12 - 6r$
23. $-3 + M = 4M - 9$
24. $14 - 3z = 13 + z$
25. $2t - 7 = 4 - t - 11$

Optional Exercises

26. $3x - 2.1 = x + 5$
27. $4A + .44 = A + .8$
28. $5r - 7 = r + 4.84$
29. $3k - 2.6 = 8.5 - 2k$
30. $3 - 7y = 2y + 8.13$
31. $x - 7 = .4x + 5$
32. $4V + .18 = V + 6.6$
33. $N - 4.2N = .8N + 3$
34. $.3n - .82 = .01n + .05$
35. $4.3 - 2.5x = 1.07x - 4.625$
36. 20% of $x = 7$
37. 25% of $x = 18$
38. $12\frac{1}{2}\%$ of $x = 85$
39. $16\frac{2}{3}\%$ of $x = 8.3$
40. $x - 50\% x = 15$
41. $x + 33\frac{1}{3}\% x = 28$

Honor Work

42. $Ax = b$
43. $mx + k = n$
44. $by - 3 = 4a$
45. $cx - a = ab$
46. $ax + c = c$
47. $3x - a = x + b$
48. $ax = bx + 20$
49. $mx - x = k$
50. $cx - 4 = b + dx$
51. $ax + bx + dx = 0$

Problems

52. Le Havre is in longitude − 2° and New York in + 73°. If the *Normandie* travels about 1° every 2 hrs., in how many hours will it travel from Le Havre to New York?

53. A balloon ascending into the stratosphere reported a temperature of − 63° at 12 mi. above the earth. If the temperature at the ground was + 57° that day, what was the average fall in temperature per mile rise?

54. A boy on the edge of a cliff throws a stone upward with a velocity of 64 ft. a sec. Using the formula $d = 64t - 16t^2$, what distance up will it be at the end of (*a*) 3 sec.? (*b*) 5 sec.? (*c*) How do you interpret the answer to *b*? (*d*) Does *d* increase or decrease as *t* increases from 4 to 5? (*e*) Can you discover for what value of *t* *d* stops increasing?

Consecutive Number Puzzles

1. Consecutive numbers are whole numbers (integers) taken in order, such as 13, 14, 15, 16. What do I add to 13 to get the next consecutive number? To get the second one? The third?

2. If n is the first of three consecutive numbers, what is the second? The third?

3. Numbers like 32, 34, 36 are called consecutive even numbers. What must I add to 32 to get the second number? The third?

4. If n is an even number, what is the next larger even number? The next larger than that one? What is the even number next smaller than n?

5. Numbers like 17, 19, 21 are called consecutive odd numbers. What must I add to 17 to get the second odd number? The third?

6. If n is an odd number, what is the next consecutive odd number? The odd number next larger than that one? The consecutive odd number next smaller than n?

7. If n is an even number, what are the next two larger odd numbers?

8. If n is an odd number, what are the next two larger even numbers?

Illustration. Find three consecutive odd numbers whose sum is 81.

Let n = the first
Then $n + 2$ = the second
$n + 4$ = the third
$n + n + 2 + n + 4 = 81$

$3n + 6 = 81$
$n = 25$
$n + 2 = 27$
$n + 4 = 29$

Check:

25, 27, and 29 are consecutive odd numbers.

$25 + 27 + 29 = 81$

9. Find three consecutive numbers whose sum is 42.

10. Find four consecutive even numbers whose sum is 60.

11. Find three consecutive odd numbers whose sum is 45.

12. Find two consecutive even numbers such that 3 times the first added to the second equals 74.

13. Find three consecutive odd numbers such that twice the smallest plus the second and the third equals 58.

14. Find three consecutive numbers such that the sum of the second and third decreased by the first is 11.

15. The sum of the first two of three consecutive numbers is 23 more than the last number. Find the numbers.

16. Paul said "Find four consecutive whole numbers whose sum is 100." John said the problem was impossible. Show which was right.

17. Can the sum of 3 consecutive numbers equal 100?

SIMPLE EQUATIONS

A short cut in solving equations—transposing.

Illustration 1. Solve: $x - 3 = 8$

To get rid of the -3, we add $+3$ to both members. Suppose we only indicate the addition.

$$\begin{array}{rcl} x - 3 & = & 8 \\ +3 & = & +3 \\ \hline x \quad & = & 8 + 3 \\ x & = & 11 \end{array}$$

In the original equation we had -3 on the left side of the equals sign. In the new equation we have $+3$ on the right side. The result is the same as if we moved the -3 past the equals sign and changed its sign from $-$ to $+$.

Illustration 2. Solve: $6x = 5x + 4$

Adding $-5x$:

$$\begin{array}{rcl} 6x & = & 5x + 4 \\ -5x & = & -5x \\ \hline 6x - 5x & = & +4 \\ x & = & 4 \end{array}$$

In the original equation we had $+ 5x$ on the right side. In the new equation we have $- 5x$ on the left side. The result is the same as if we moved the $+ 5x$ past the equals sign and changed its sign from $+$ to $-$.

Rule. A term may be moved from one side of the equals sign to the other side provided its sign is changed.

We say that the term is *transposed* from one side of the equation to the other. In solving an equation, all terms containing the unknown letter should be collected on one side of the equation, usually the left, and all other terms should be collected on the other side. Transpose all those terms that are not on the proper side. Terms that are not transposed, however, do not have their signs changed.

Illustration. Solve: $5x - 1 - 3x = 2x + 4 - x$

Transposing: $5x - 3x - 2x + x = +4 + 1$

$x = 5$

Check: $25 - 1 - 15 \mid 10 + 4 - 5$

$9 = 9$

The $- 1$ is transposed to the right side and becomes $+ 1$; the $+ 2x$ is transposed to the left side and becomes $- 2x$; the $- x$ is transposed to the left side and becomes $+ x$.

Exercises

Solve the following equations, and check your answers:

1. $3x - 5 = x - 1$

2. $4x + 14 = 4 - x$

3. $6y - 4 = 3y + 8$

4. $3z + 8 = 26 - 3z$

5. $5t - 14 = 2 + t$

6. $7x + 6 = 2x - 4$

7. $2 - 5r = 2r + 16$

8. $3x + 4 = 9 + 4x$

9. $8s + 3 = 13 - 2s$

10. $9 - 3k = k + 1$

11. $4x - x = 7 + x$

12. $4 - 2y = y + 5$

13. $3m - 2 = 7m - 6m$

14. $3 - 2(x + 4) = -7$

15. $5(x - 2) = 3x - 4$

16. $3(k + 1) - 5 = k$

17. $4(m - 2) - 7 = 2m$

18. $3z + 4(1 - z) = 2z$

19. $4 - 3(x - 1) + 7 = 0$

20. $2(p - 3) = -3(p + 4)$

Review Exercises

Solve and check:

1. $5x - 3 = 2x + 12$
2. $4x + 5 = x + 11$
3. $7y - 11 = 2y - 6$
4. $2a - 4 = 11 - 3a$
5. $8 - 2m = 6m - 12$
6. $4H + 9 = -9 - 5H$
7. $R - 13 = 5R - 8$
8. $w_1 + 3 = 5 - w_1$
9. $4m + 13 = 12 - m$
10. $n + 5 = 4n + 5$
11. $5x - 3 = x + 4$
12. $x - 4k = 13 - 4k$
13. $2.3h + .16 = 1.68 - 1.7h$
14. $2\frac{1}{2}k + 3 = 1\frac{1}{2}k + 18$
15. $\frac{P}{2} + 1 = \frac{P}{3} + 7$
16. $\frac{3e}{5} - \frac{e}{6} = e - 34$
17. $2r - \frac{r + 7}{8} = 1$
18. $\frac{3x - 6}{4} - \frac{2x - 8}{3} = 2$

Multiply:

19. $3(5x - 8)$
20. $5x(1 - n)$
21. $3xy(-2x)$
22. $-4m^3(+5m)$
23. $-y(x - y)$
24. $-5(-2)(4)$
25. $pqr(pr)$
26. $4x^4(2x^2)$
27. $(-3x^2)^3$
28. $(2m - r)^2$
29. $3x(1.3x^2)$
30. $(-1)^4$

Divide:

31. $\frac{a}{a}$
32. $\frac{3h}{h}$
33. $\frac{6r}{3r}$
34. $\frac{8a^8}{4a^4}$
35. $\frac{m}{-m}$
36. $\frac{6b^3 - 9b^4 + 12b^2}{3b^2}$
37. $\frac{14r^2 - 7r^5 + 7}{7}$
38. $\frac{12x^3y^5 - 9x^2y^4 - 15x^6y^2}{3xy^2}$
39. $\frac{5m^2 - 5mn + 10n}{-5}$
40. $\frac{x - xy - x^2}{-x}$

Subtract:

41. $\begin{array}{r} -4m^3 \\ \underline{+3m^3} \end{array}$
42. $\begin{array}{r} -5r \\ \underline{-5r} \end{array}$
43. $\begin{array}{r} k \\ \underline{4k} \end{array}$
44. $\begin{array}{r} -3x^2y^4 \\ \underline{-7x^2y^4} \end{array}$
45. $\begin{array}{r} 0 \\ \underline{-3a} \end{array}$

Combine like terms:

46. $-3x + 7 + 4x - 2 - 3x + 4 + 3x - 5$

47. $5m - (-3m) + (-3m) - (+5m)$

48. $3x^2y - 2xy^2 + 5xy^2 - 4x^2y$

Solve:

49. $5(x - 2) - 3(2 + x) + 7 = 2x + 4(x - 3) - 13$

50. $3x(1 - 2x) - 4(x - 3) = 6x(5 - x) - 19$

51. If $y = 2x - 1$, make a table of values of y for $x = 0, 1, 2,$ 3, 4, and 5.

52. Two sides of a triangle are each x inches long and the third side is 3 more than their sum. What is the perimeter?

53. In $K = 2a - 5(n - 1)$, find the value of K when $a = 8$ and $n = 3$.

54. A team won a games and lost b games. What fraction of its games did it win?

55. If m and n are two numbers, what is their average?

56. If $y = \frac{12}{x}$, how does y change when x grows larger?

57. Remove the parentheses and collect terms:

$$5m - 2a - 3(a - 3m) + 2(m + a)$$

58. Find the value of r in $r = \frac{gs}{g + s}$ when $g = 8$ and $s = 12$.

HISTORICAL NOTE ON THE SHORTHAND OF ALGEBRA: It is surprising how long it took the world to develop a good symbolism for algebra. Although geometry was well known to the Greeks, they never succeeded in working out even a satisfactory number system. All early writers except Diophantus used the full word or an abbreviation for it, so that their equations were more like rules than modern equations. Diophantus (275 A.D.) used many symbols but his work had little influence on later mathematics, for all through medieval times, no satisfactory symbols were used for even the operations of arithmetic. Plus (+) and minus (−) seem to have been first used in Widman's arithmetic (1489) in Germany. The multiplication sign (×) appeared in England about 1600. Our equality sign (=) was first used by an Englishman, Recorde in 1557. Before his time the whole word or an abbreviation of it had been used. Vieta (1590), a French mathematician, used letters to represent both known and unknown numbers. His is the earliest book to resemble a modern text in algebra.

59. A man earns e dollars a month and spends s dollars a month. How much does he save (*a*) in 1 mo.? (*b*) In 2 yrs.?

60. Factor:

(*a*) $5x - 5y$
(*b*) $am + 2an$
(*c*) $6x^3y - 4x^2y^2$
(*d*) $3b^3 - 3b^2$
(*e*) $ax - bx - cx$
(*f*) $10p^2 - 15p^3 + 5p$

61. Find the value of:

(*a*) $79.34 \times 834 - 79.34 \times 833$
(*b*) $27 \times 7\frac{1}{4} + 5 \times 7\frac{1}{4}$

62. What per cent of 852 is 355?

63. Make a graph of $v = 12 - 2t$ from $t = 0$ to $t = 6$.

64. A room is 15 ft. long, 12 ft. wide, and 9 ft. high. How many square feet of plaster are required?

Formula: $N = 2lh + 2wh + wl$

65. The formula for finding the number of centimeters equal to a certain number of inches is $c = 2.54i$.

(*a*) Make a table of values of c for values of i from 0 to 10.

(*b*) Make a graph from your table.

(*c*) Find the number of centimeters corresponding to $7\frac{1}{2}$ in.

(*d*) Find the number of inches corresponding to 8 cm. To 23 cm.

(*e*) Does the number of centimeters increase or decrease as the number of inches increases? If you double the number of centimeters, is the number of inches doubled?

Test in Multiplication

Find the product of:

1. $-3a(+4)$
2. $(5b)(-2b)$
3. $-3m(-2m)$
4. $(+2a)(+6a)$
5. $5b^2(2b^3)$
6. $-6r^3(-4r^5)$
7. $-2v(-w)$
8. $a^3(-a)$
9. $4x^4(4x^4)$
10. $-3a^2b(-2b^3)$
11. $-(xy)(-xz)$
12. $(+3m)(-5m^3y)$
13. $(5ab^2c^3)(3a^3bc^4)$
14. $2x(-5xy)(+3yz)$
15. $2x^2(3x^3)(4x^4)$
16. $-(-3p)(-5p^2)$
17. $(-1)^2(-2)^3(+1)^6$
18. $(2a^2)(-2a)(2a^3)$
19. $4863(5976)(0)$
20. $-(-1)(-2)(-3)$

Test in Division

Find the quotient:

1. $\frac{5x}{x}$
2. $\frac{7b}{7}$
3. $\frac{-2m}{2}$
4. $\frac{-6c^2}{3c^2}$
5. $\frac{4a^2}{4a}$
6. $\frac{-4x^4}{-2x^2}$
7. $\frac{9m^3}{9m^3}$
8. $\frac{-5x}{5x}$
9. $\frac{-3x^3y}{-3x^2y}$
10. $\frac{20x^7y}{-4xy}$
11. $\frac{3x-3}{3}$
12. $\frac{8x^2-2x}{2x}$
13. $\frac{9a^3-6ab}{-3a}$
14. $\frac{5-10s^3}{-5}$
15. $\frac{7b^3-b}{b}$
16. $\frac{4x^3y-2xy^2}{xy}$
17. $\frac{3m^2n-n}{-n}$
18. $\frac{-5x^3+3ax}{-x}$

Test in Factoring

Factor:

1. $6m + 6r$
2. $5h - 15k$
3. $m^3 - 2m^2$
4. $-8x^4 + 4x^3$
5. $\pi r^2 + \pi rh$
6. $\frac{1}{2}hm + \frac{1}{2}hn - \frac{1}{2}hp$
7. $P + Prt$
8. $72.96 \times 813 - 72.96 \times 613$
9. $547 \times 415 - 536 \times 415$
10. $34 \times 55 \times 2 - 34 \times 45 \times 2$

Test in Solving Equations

Solve:

1. $x + 5 = 8$
2. $4y = 20$
3. $5a + 7 = 22$
4. $3b + 8 = 5$
5. $7 - 2c = 3$
6. $2x = 5x - 12$
7. $\frac{x}{4} = 9$
8. $\frac{3x}{5} = 21$
9. $\frac{m}{3} - 1 = 5$
10. $4x + 5 = 10 - x$
11. $3.4x = 10.2$
12. $5x = 3.9x + 33$
13. $.3x - 7 = 2$
14. $4(2x - 7) = x + 7$
15. $8x - (-3 + 5x) = 4$
16. $\frac{x}{5} + \frac{x}{2} = 14$
17. $\frac{3x}{2} - \frac{x}{6} = 24$
18. $\frac{2x+1}{5} - \frac{x-1}{6} = 2$

Chapter 8

PROBLEMS

ONE of the greatest values of algebra to you is the ability it gives you to solve problems. In life outside school, unless you are in certain lines of work, you will find very few equations ready-made for you to solve, but you will meet many problems that you must solve in some way, and usually the algebraic method is the easiest way. Then too, problems are a better test of your ability to think than any other part of algebra. In this chapter you will practise solving easy problems.

Algebraic Shorthand

Write the equations for these statements on a separate sheet as they are being dictated by the teacher. Then solve your equations:

1. When 3 is added to 4 times a number, the result is 31.

2. If 5 times a number is diminished by 13, the result is the same as if twice the number were increased by 20.

3. Seven more than 3 times a number is 34.

4. If a number is divided by 3 and then 4 is added, the result is 1 more than $\frac{1}{2}$ the number.

5. If a number is divided by 4 and then 8 is subtracted, the result is 1.

6. Twice a number decreased by 9 equals 41.

7. Six times a number exceeds twice the number by 24.

8. I am thinking of a number. If I multiply it by 4 and then add 3, the result will be 23.

9. Twice a number exceeds 10 as much as 17 exceeds the number.

10. If 3 is subtracted from $\frac{1}{5}$ of a number, the result is 4.

11. Twice a number increased by 52 is 20 more than 6 times the number.

12. If 3 is added to 7 times a certain number, the result is 18 more than 4 times the number.

13. Five times a number equals 18 diminished by the number.

14. Find a number whose double divided by 3 is 10

15. Four less than 5 times a number is 26.

Problems Having More Than One Unknown

Problem. One number is 7 more than twice another number, and their sum is 22. Find the numbers.

Notice that there are two unknown numbers and two parts to the statement. Let some letter stand for one of the numbers, usually the smaller number. Use one of the parts of the statement for representing the other number. Then use the other part for making the equation.

Solution. Let n = the smaller number
Then $2n + 7$ = the larger number (The first part)
$n + 2n + 7 = 22$ (The second part)
$3n + 7 = 22$
$3n = 15$
$n = 5$
$2n + 7 = 17$

Check: 17 is 7 more than twice 5. The sum of 5 and 17 is 22.

1. Find two numbers if one is 5 times the other and their sum is 42.

2. One number is 3 times another, and their difference is 11. Find the numbers.

3. The greater number exceeds the smaller by 9, and their sum is 21. Find the numbers.

4. Find the dimensions of a lot if the length is 20 ft. more than the width and the fence around it is 280 ft. long.

5. One number exceeds another number by 6, and their sum is 40.

6. One number is 3 more than another, and their sum is 9.

7. One number is 3 times another, and their sum is 20.

8. Separate 33 into two parts so that the larger is twice the smaller.

9. Separate 15 into two parts so that the larger is 1 greater than the smaller.

10. One number is twice another. If the larger is diminished by 10, the result is 2 greater than the smaller.

11. The difference of two numbers is 8, and twice the smaller is 6 more than the larger.

12. If one of two equal numbers is increased by 7 and the other is diminished by 2, their sum becomes 25.

13. The length of a city lot is 75 ft. more than the width, and its perimeter is 250 ft. Find its dimensions.

14. A book and a pencil cost $1.10, and the book cost $1 more than the pencil cost. What did the pencil cost?

15. A contractor ordered 460 lbs. of concrete containing 4 times as much sand as cement. How many pounds of cement will be needed?

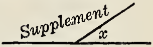
16. Find two numbers whose sum is 91, if the greater is 6 times the smaller.

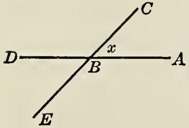
17. If 3 is added to $\frac{1}{5}$ of a number, the result is $\frac{2}{5}$ of the number. What is the number?

18. The larger of two numbers exceeds twice the smaller by 6. The larger is also 1 more than 3 times the smaller. Find the numbers.

VERTICAL ANGLES

1. An angle and its supplement together contain 180°. How many degrees has the supplement if the angle has 50°? 90°? 160°? x°?





2. If $\angle x$ is growing larger, how is its supplement changing?

3. When $\angle x$ has become as large as its supplement, how many degrees has each?

4. There are x° in $\angle ABC$. How many degrees are there in its supplement $\angle CBD$?

5. Since CBE is also a straight line, $\angle ABE$ is a supple-

ment of ∠ABC. If ∠ABC has $x°$, how many degrees are there in ∠ABE?

6. How does the number of degrees in ∠ABE compare with that in ∠CBD? If x grew larger, would these angles still remain equal?

7. But ∠s CBD and ABE are vertical angles. Are vertical angles always equal?

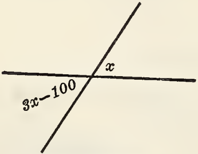
In these exercises we learned that ∠s CBD and ABE are each $180 - x$, so no matter what number x stands for, they are always equal.

Vertical angles are equal.

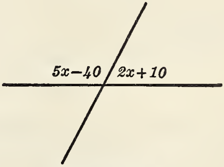
> HISTORICAL NOTE: That vertical angles are equal was probably the first proposition in geometry ever proved. By proving it and a few other equally obvious propositions before anyone else thought to do so, Thales, about 560 B.C., became the founder of demonstrative geometry.

The Equation in Geometry

1. Find x if one of two vertical angles has $x°$ and the other has $3x - 100$ degrees.



Vertical Angles

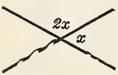


Supplementary Angles

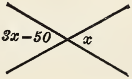
2. Find x if one of two supplementary angles has $2x + 10$ degrees and the other has $5x - 40$ degrees.

Find x:

3.



4.



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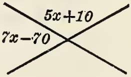
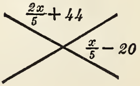
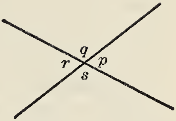


6.



Find x:



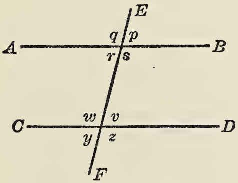
7. If $p = x + 10$ and $r = 2x - 20$
8. If $p = 5x - 80$ and $q = 80 - x$
9. If $p = 2x$ and $r = 200 - 3x$
10. If $q = \frac{x}{2}$ and $s = x - 60$
11. If $p = \frac{x}{5}$ and $q = \frac{x}{4}$
12. If $r = 2x - 50$ and $q = \frac{7x + 110}{4}$
13. If $s = \frac{8x}{3} + 18$ and $q = \frac{5x}{6} + 40$
14. If p is a right angle and $r = \frac{4x}{3} - 50$

15. $\angle p$ is growing larger. What change is taking place in $\angle r$? In $\angle q$? In $\angle s$?

16. As p grows from 0° through 90° to 180°, describe the change in q giving the number of degrees for the three sizes mentioned.

PARALLEL LINES

The lines *AB* and *CD* both lie on this sheet of paper and yet would never meet no matter how far they were extended. They are crossed by a third line called a transversal. Do any of the angles look equal? Name the angles that appear equal to *p*. Which angles at the line *CD* are equal to *r*? To *s*? Is there an angle on *CD* equal to *q*?



The pair of angles *r* and *v* and the pair *s* and *w* are called ***alternate interior angles.*** Can you discover why they are called interior angles? Why alternate angles?

The pair of angles *q* and *w*, or *p* and *v*, or *s* and *z*, or *r* and *y*, are called ***corresponding angles.*** In what way do they correspond?

Are corresponding angles on the same side or on opposite sides of the two lines? Are alternate interior angles on the same side or on opposite sides of these lines? Which kind are on the same side of the transversal and which on opposite sides?

Two *lines are parallel* (||) if they cross a transversal at the same angle, that is, if the corresponding angles are equal. If $\angle p$ equals $\angle v$, then AB is parallel to CD.

If a pair of corresponding angles are equal, do you think that a pair of alternate interior angles are necessarily equal? Suppose $p = v$, does r necessarily equal v? What is the name for the pair of angles p and r? Are they equal? If $p = v$ and $p = r$, does $v = r$? If lines are parallel, are alternate interior angles necessarily equal? If alternate interior angles are equal, are corresponding angles necessarily equal? Are the lines necessarily parallel?

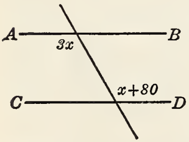
If two lines are parallel:

1. **Alternate interior angles are equal.**
2. **Corresponding angles are equal.**

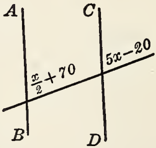
Exercises

In these exercises AB is parallel to CD.
Find the value of x, and check:

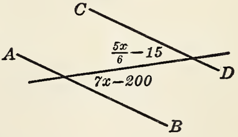
1.



2.



3.

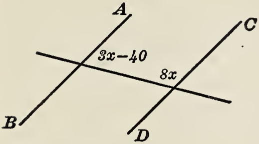


4. If $r = 70°$, how many degrees are there in v? In s? In w? What is the sum of s and v? Of r and w? (See page 165.)

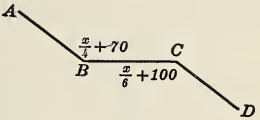
5. If $r = k°$, how many degrees are there in v? In s? In w? What is the sum of s and v? Of r and w? Is the sum of a pair of interior angles on the same side of the transversal always the same, no matter what value k has? What is this sum?

Find the value of x:

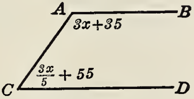
6.


A
C
$3x-40$
$8x$
B
D

7.


A
$\frac{x}{4}+70$
C
B
$\frac{x}{6}+100$
D

8.


A
B
$3x+35$
$\frac{3x}{5}+55$
C
D

Find the value of x:

9. If $r = 4x - 30$ and $v = x + 30$

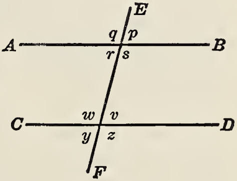
10. If $s = 3x - 10$ and $v = x - 10$

11. If $q = \frac{x-10}{3}$ and $s = \frac{x+20}{4}$

12. If $r = 7x + 3$ and $v = \frac{5x}{2} + 75$

13. If $r = \frac{x}{5}$ and $w = \frac{2x}{3} + 50$

14. If $p = \frac{x-10}{3}$ and $v = \frac{x}{2} - 28$


E
q
p
A
B
r
s
w
v
C
D
y
z
F

In these exercises show whether AB and CD are parallel:

15. $r = 50°$ and $v = 50°$

16. $s = 140°$ and $z = 140°$

17. $q = 110°$ and $v = 80°$

18. $q = 120°$ and $w = 60°$

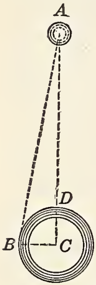
19. If line *EF* is turning around so that *p* is growing larger, what change is taking place in *r*? In *q*? In *s*? In *v*? In *w*?

20. If *p* becomes a right angle, what does *w* become?

21. If *AB* is not parallel to *CD*, (*a*) Does $p = v$? (*b*) Does $p = r$? (*c*) Does $r = v$?

THE SUM OF THE ANGLES OF A TRIANGLE

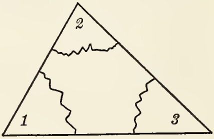
To find the distance to the moon *A*, an astronomer at *B* on the earth must find the number of degrees in ∠*A* at the moon. Do you think he can determine this from the two angles on the earth, ∠*B* and ∠*C*? Can you discover any relation between the angles *A*, *B*, and *C*? Do you think their sum is always the same? Would a small triangle have just as many degrees in the sum of its angles as a large triangle has? How great does that sum appear to be?

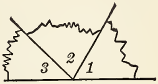


1. Draw a large triangle and a small triangle, and measure the angles with your protractor. What is the sum in the large triangle? In the small triangle?

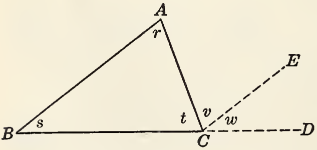
2. Draw two triangles of very different shape, and measure their angles. Is the sum always the same?

3. Draw a large triangle, and cut it out. Now tear off the three angles, and fit them together as shown here. Is the sum a straight angle?





4. In the triangle *ABC* the base *BC* is extended to *D*, and *CE* is drawn parallel to *BA*. What is the sum of $t + v + w$? If $CE \parallel BA$, what kind of angles are *v* and *r*? *w* and *s*? How does the number of degrees in *v* compare with that in *r*? In *w* compare with that in *s*? How does the sum $r + s + t$ compare with the sum $t + v + w$? How many degrees then is the sum $r + s + t$?



The sum of the angles of a triangle is 180°.

Exercises

Find the number of degrees in $\angle C$:

1. If $A = 38^\circ$ and $B = 74^\circ$

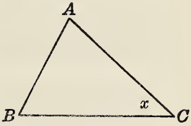
2. If $B = 48^\circ$ and $A = C$

3. If all three angles are equal

4. If A is a right angle, and B is twice as large as C

5. If A is a right angle and $B = C$

6. If $C = x$, $A = 2x$, and $B = x + 20$



7. If B is 10° larger than C, and A equals the sum of B and C

8. If B is 56° larger than C, and A is twice as large as C

9. If B is twice C, and A is 3 times C

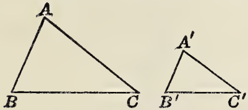
10. If B is twice C, and A equals the sum of B and C

11. If A is a right angle and C is growing larger, how is B changing?

12. If triangle ABC grows larger, do the angles grow larger, grow smaller, or remain the same? Consider the cases (*a*) the triangle remains the same shape; (*b*) the triangle changes its shape.

Thought Questions

1. If two angles of one triangle equal two angles of another triangle, what do you know about the third angles of these triangles?



2. (*a*) If $\angle A = 80^\circ$ and $\angle B = 70^\circ$, find the size of $\angle C$.

(*b*) If $\angle A' = 80^\circ$ and $\angle B' = 70^\circ$, find the size of $\angle C'$.

3. (*a*) If $\angle A = x^\circ$ and $\angle B = y^\circ$, find the size of $\angle C$.

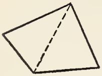
(*b*) If $\angle A' = x^\circ$ and $\angle B' = y^\circ$, find the size of $\angle C'$.

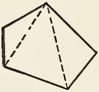
(*c*) How do $\angle$s C and C' compare in size?

Investigating the polygon. A figure made of straight lines enclosing a part of the plane is called a ***polygon.***

1. How many sides has a triangle? How many straight angles are there in the sum of its angles?

2. How many straight angles are there in the sum of the angles of a four-sided polygon? Of a five-sided polygon? Of a six-sided polygon?





3. Complete this table, where s is the number of straight angles and n is the number of sides the polygon has:

n	3	4	5	6	7	8	9	10	....	n
s	1	2	3						....	

4. From this table make a formula beginning $s =$

5. Make a graph from the formula you made for Exercise 4.

Algebraic Expression

1. Find 6% of \$11, of \$800, of P dollars.

2. Find 5% of \$100 for 1 yr., for 3 yrs., for t yrs.

3. Find the proceeds (amount left) if a discount of 6% is deducted from \$100, from \$400, from P dollars.

4. An agent receives a commission of 10% on sales. How much is the commission if the sales amounted to \$800? To S dollars?

When r is used to express the rate, it includes the per cent; that is, if the rate were 4%, r would be .04.

5. If the rate of interest is r, find the interest on \$100 for 2 yrs., for t yrs.

6. Find the interest on P dollars at rate r for 7 yrs., for t yrs.

Express these sentences in algebra shorthand:

7. 60 is 4% of what number?

8. What per cent of 90 is 81?

9. What number is 8% of 320?

Answer these questions:

10. 15 exceeds 12 by what per cent of 12?

11. x exceeds y by what per cent of y?

12. What is the selling price if an article that cost \$40 sells at a gain of 20% of the cost? That cost c dollars sells at a gain of 20% of the cost?

13. What is the selling price if an article that cost $24 sells at a loss of 25% of the cost? That cost c dollars sells at a loss of 25% of the cost?

14. What fraction of the cost does a dealer make:

(*a*) If he buys chairs at $12 each and sells them at $16 each?

(*b*) If he buys chairs at $12 each and sells them at s dollars each?

(*c*) If he buys them at c dollars each and sells them at s dollars each?

15. What number increased by $33\frac{1}{3}\%$ of itself equals 160?

16. What number decreased by 25% of itself equals 33?

The Equation in Business

How to solve percentage problems.

Illustration. An author receives a royalty of 8% on the sale of his book. If the publishers sent him a check for $1624, what was the amount of the sales?

Let x = amt. of the sales

$.08\,x$ = no. of dollars in his royalty

$$.08\,x = 1624$$

$$x = \frac{1624}{.08}$$

$$= 20{,}300$$

Check:
8% of 20,300 = 1624

1. Mr. Walsh sells goods at a profit of 20% of the cost. He uses the formula $s = c + .20\,c$. He never uses the per cent sign (%) in his formulas. Find s if $c = \$1.40$.

2. Mr. Houghton sells used cars at 40% below the cost. Make a formula for s. Find s when $c = \$680$.

3. A merchant makes a profit of 15% of his sales. How much must he sell this month to make $375?

4. Nathan Goldberg, a clothier, wishes to put a suit on sale at 70% of the marked price. At what price must he mark it so that he can sell it for $35?

5. A grocer makes a profit of $420 on the sale of $4620 worth of groceries. What per cent of his sales does he gain?

6. A dealer bought shoes for $6.30 a pair. At what price must he sell them to make 25% of the selling price?

7. Mr. Campbell made a profit of $22.50 on the sale of a machine. He says that this was only 15% of the cost. What was the cost?

8. The Economy Jewelry Store wishes to sell a clock for $8 after giving the purchaser a discount of $33\frac{1}{3}$% of the marked price for cash. At what price must they mark the clock?

9. Mr. Martin must borrow $343 from his bank for 4 mo. If the bank deducts a discount of 2% of the face of the note, for what amount must he write the note?

10. The Harvey Marketing Company received a check for $3230 from their broker with the statement that he had deducted his commission of 5% of the amount of the sales. What was the amount of the sales?

11. For what amount must I write a promissory note if I need $2910, but must leave the bank 3% discount on the face of the note?

12. Mr. Hammond paid an income tax of $741, 3% of his income. What was his income?

13. Mr. Haddon argues that the city should not object to spending $84,500 for the new school building as this is only $2\frac{1}{2}$% of the taxable value of the property in the city. If this is true, what must be the value of the taxable property?

14. An importer bought goods in Europe, but had to pay an import duty of 30% ad valorem (of the cost). If the final cost of the goods was $2678, what did they cost in Europe?

15. What change takes place in a merchant's profits if:

(*a*) The cost of the goods increases, but the selling price remains unchanged

(*b*) The cost increases, but the per cent of profit on the cost remains unchanged

(*c*) The selling price increases, but the cost remains unchanged

(*d*) The cost and selling price increase by the same amount

PROBLEM ANALYSIS

How to analyze a problem by the box method. Solving problems is a game. As in any game, you will succeed best if you know the rules and follow them.

Rules of the game of solving problems. 1. First look at the question. Then put x in the box that represents the quantity, or one of the quantities, you are asked to find.

2. Fill the other boxes in the same column from data given in the problem.

3. If there are three columns, find data in the problem from which you can fill a second column.

4. Fill the remaining column of boxes (not always the right-hand column) from the quantities in the other boxes. Do not use information from the problem in filling these boxes.

5. Your equation will be made in terms of the subject of the last boxes filled. Read the problem again carefully to see what it tells about the subject of these last boxes. Make your equation using only this information and the expressions in the last column of boxes filled. Do not mix columns in your equation.

Illustration. Widow Jones has been left \$20,000. She wishes to invest it safely at 4%, but as she needs an income of \$1040 a yr., she must put part of it in a less safe investment at 7%. How much can she invest at 4%?

1. The question is "How much can she invest at 4%?" So let x be the principal at 4%.

Principal	*Rate*	*Income*
x	.04	
$20{,}000 - x$	.07	

2. To fill the box under x, note that she has 20,000 to invest, so the rest of it, $20{,}000 - x$, is to be invested at 7%.

Principal	*Rate*	*Income*
x	.04	$.04\,x$
$20{,}000 - x$	.07	$.07(20{,}000 - x)$

3. The rate column is filled from the 4% and 7% given in the problem.

4. The income column is filled from the product

of the principal and rate. Do not take anything from the wording of the problem for this column.

5. Now the equation must be made in terms of income, for the income boxes were the last filled. So look for income in the problem. It says, "She needs an income of $1040." Therefore:

$$.04\,x + .07(20000 - x) = 1040$$

Solving:

$$\begin{aligned} .04\,x + 1400 - .07\,x &= 1040 \\ 4\,x + 140000 - 7\,x &= 104000 \\ 4\,x - 7\,x &= 104000 - 140000 \\ -3\,x &= -36000 \\ x &= 12000 \end{aligned}$$

Check:

$$\begin{array}{rcr} 12000 \times 4\% &=& 480 \\ \underline{8000} \times 7\% &=& \underline{560} \\ 20000 & & 1040 \end{array}$$

The Equation in Investment Problems

1. I have $8000 from which I wish to receive an income of $380 a yr. If I invest as much as possible in 4% bonds and the rest in 6% stock, how much must I invest in the stock?

2. Henry Adams invested $3000, partly at 5% and partly at 6%. He says that he receives $18 a year more from the 5% investment than from the 6% one. How much has he at each rate?

3. Mr. Houghton has $4000 in a checking account that pays him only 2%. He decides to draw out and deposit in a savings bank that pays $4\frac{1}{2}$% enough of it so that his interest will amount to $150. How much should he put in the savings bank?

4. Mrs. Wilson has two investments, one paying 5%, and the other, $1000 larger than the first, paying 6%. Her income is $258. How much money has she at each rate?

5. In his will a man directs that his daughter shall receive enough of his estate to furnish her with an income of $1400 a yr., $\frac{2}{3}$ of her estate being in 5% bonds and the remainder in 4% bonds. How much of each should she receive?

6. If you invest $1000 at 4%, how much would you have to invest at 6% so that you would receive $5\frac{1}{2}$% of your total investment?

7. Robert Smith left a hardware company because they paid him only 2% commission on his sales of $8000 and went to work for a farm equipment company that offered him a commission of 5% on his sales. How much must he sell there the remainder of the year so that his total income may equal 3% of his total sales for the year?

8. What amount of money must a father invest at 6% simple interest so that his son will receive $2832 3 yrs. later for college expenses?

9. Mr. Arnold's mortgage for $924 will fall due in just 2 yrs. How much money should he put at interest at 5% so that it will amount to enough to pay the mortgage?

10. Chester Brown invested a sum of money at 5% and 3 times as much at 4%. The second investment pays $140 a yr. more than the first. How much has he at each rate?

11. Mr. Banker lent $3000 at 4%. How much additional must he lend at 6% so that the total income will be $4\frac{1}{2}$% of the total loan?

Algebraic Expression

1. An automobile travels 35 mi. an hr. How far will it go in 6 hrs.? In 8 hrs.? In t hrs.?

2. How far can a boy row in t hrs. if he can row 2 mi. an hr.? r mi. an hr.?

3. Find a man's rate if he runs 120 yds. in 15 sec., in 12 sec., in t sec.

4. How long will it take to go 100 mi. at 8 mi. an hr.? At t mi. an hr.?

5. How long will it take to go m mi. at 10 mi. an hr.? At t mi. an hr.?

6. If you know the rate and the time, what would you do with them to find the distance? Write a formula for this relation using r, d, and t.

7. If you know the distance and the time, what would you do with them to find the rate? Write a formula for this relation.

8. If you know the distance and the rate, what would you do with them to find the time? Write a formula for this relation.

9. Can the formulas in Exercises 7 and 8 be derived from that in Exercise 6? Show this by solving $rt = d$ for the proper letter.

Illustration. A local passenger train starts east from Denver at the average rate of 30 mi. an hr. Two hrs. later a fast express leaves Denver over the same track at a speed of 50 mi. an hr.

In how many hours after the express leaves must the train dispatcher arrange for the express to pass the local?

1. The question reads " In how many hours after the express leaves." Let x represent this number.

2. Now fill the box in the same column with the one filled. Here the local train started 2 hrs. earlier, so its time is $x + 2$.

3. To fill a second column of boxes, notice that the rates of the express and local are 50 and 30 mi. an hr.

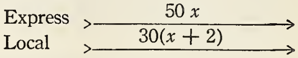
	Rate	*Time*	*Distance*
Express. .	50	x	
Local. . . .	30	$x + 2$	

4. Using $rt = d$, fill the last boxes without looking at the problem.

	Rate	*Time*	*Distance*
Express. .	50	x	$50x$
Local. . .	30	$x + 2$	$30(x + 2)$

5. The equation is made in terms of distance. So reading the problem again, we see that the distances are equal. Then:

$$50x = 30(x + 2)$$
$$x = 3$$



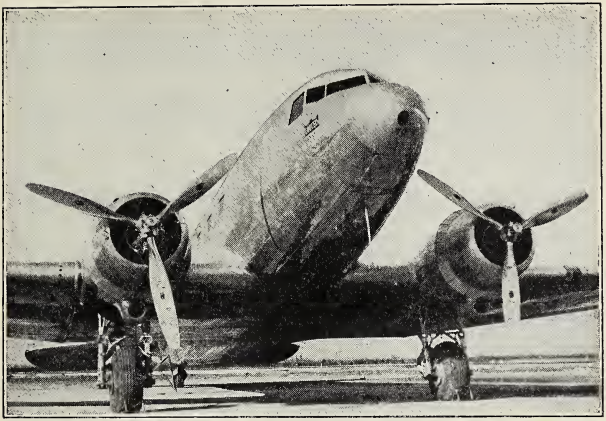
The Equation in Motion Problems

1. George Brown leaves Atlanta at noon traveling south at the rate of 30 mi. an hr. At 2 P.M. another car leaves Atlanta with orders to overtake Brown by 5 P.M. At what rate must this car travel?

2. A slow train leaves Chicago for New York at 10:30 A.M. traveling at an average speed of 40 mi. an hr. Two hrs. later the Twentieth Century Limited leaves Chicago at an average speed of 60 mi. an hr. How far from Chicago will the Twentieth Century pass the other train?

3. An airplane leaves Oakland for the East traveling 180 mi. an hr. At the same time another airplane leaves Salt Lake City, 600 mi. farther east, and travels toward Oakland at the rate of 120 mi. an hr. In how many hours will they pass?

4. A train and a bus leave a town at the same time and travel in opposite directions. The train travels twice as fast as the bus, and in 4 hrs. they are 252 mi. apart. What are their rates?



United Air Lines Photo.

A MODERN PASSENGER PLANE

With two 1150 horse-power motors, this twin-engined Douglas transport can take off in 980 ft. and climb to 8500 ft. with only one motor operating, and to 23,000 ft. with both motors running. These planes have a high speed of 220 mi. per hr. and fly from coast to coast in 15 hrs.

5. Harold Green set out to walk home from college, a distance of 23 mi., at 4 mi. an hr., but a car picked him up after he had gone part way, and he finished his trip at the rate of 15 mi. an hr., arriving home in 3 hrs. from the time he started. How far did he walk?

6. Robert can run 12 ft. a sec. John can give him a start of 60 ft. and overtake him in 15 sec. How fast can John run?

7. George rowed down the river at the rate of 5 mi. an hr. and returned at 2 mi. an hr. How far did he go if the trip took 3 hrs.?

8. As I have to wait in St. Louis 3 hrs. for a connecting train, I walk from the station at the rate of 4 mi. an hr. How far may I go if I can ride back at the rate of 12 mi. an hr.? If I had to wait 6 hrs. for the train instead of 3 hrs., would the distance I might go be doubled?

9. A train leaves St. Albans for White River Junction averaging 40 mi. an hr. The railroad wishes to start a train from

White River Junction for St. Albans at the same time and have them pass each other 75 mi. from St. Albans. If these towns are 120 mi. apart, what speed must the proposed train average?

10. If, in Exercise 9, the proposed train were to travel at a speed of 50 mi. an hr., how far from St. Albans would they meet?

The Equation for the Merchant: Mixture Problems

Class Exercises

Illustration. Mr. Rowe has 100 lbs. of 60¢ coffee that does not sell easily. He wishes to mix it with 30¢ coffee to make a mixture worth 40¢ a lb. How many pounds of the 30¢ coffee must he use?

Kind	*Amount*	*Value*
30¢	x	$30x$
60¢	100	6000
40¢	$100 + x$	$40(100 + x)$

$$\begin{aligned} 30x + 6000 &= 40(100 + x) \\ 30x + 6000 &= 4000 + 40x \\ x &= 200 \end{aligned}$$

1. The H and A Company has tea that should sell at 70¢ a lb. With how many pounds of 30¢ tea should they mix 50 lbs. of it to make a mixture worth 40¢ a lb.?

2. In order to make a mixture of nuts to sell at 60¢ a lb., how many pounds of 40¢ nuts should be mixed with 200 lbs. of 70¢ nuts?

3. A druggist wishes to make up 200 lbs. of candy to sell for 70¢ a lb. He has some candy worth 80¢ a lb. and some worth 50¢ a lb. How much of each should he take?

4. A paint store has white lead paint worth $4 a gal. and barytes paint worth $1.50 a gal. How much of each should be used to make 50 gal. of a paint worth $2.50 a gal.?

5. The Excelsior Feed Co. wishes to make 1200 lbs. of a poultry food to sell at $3 for 100 lbs. It has an oat food worth

$4 for 100 lbs. and a scratch food worth $1 for 100 lbs. How much of each should be used?

6. H. G. Taylor has 60 lbs. of coffee worth 65¢ a lb. How much coffee worth 40¢ a lb. should he mix with it to make a mixture worth 50¢ a lb.?

7. A farmer has 300 lbs. of maple sugar worth 60¢ a lb. A city dealer will pay only 35¢ a lb. How much granulated sugar worth 5¢ a lb. must he mix with his maple sugar to make a product that he can sell for 35¢ a lb.?

8. Corn syrup is selling for 10¢ a qt. and maple syrup for 30¢ a qt. How much of each should a dealer use to make 40 qts. of a maple-flavored syrup to sell at 15¢ a qt.?

Optional Exercises

Illustration. The druggist has a 40% solution of silver nitrate, but the doctor's prescription calls for a 10% solution. How many ounces of distilled water must he add to 3 oz. of the solution to reduce it to the required strength?

Let x = amt. of water added

	Amount of Solution	%	*Amount of Silver Nitrate*
First solution....	3	.40	.40(3)
Material added..	x	0	0
Second solution..	$3 + x$	.10	$.10(3 + x)$

Since the amount of silver nitrate has not changed, the equation is:

$$\begin{aligned} .40(3) &= .10(3 + x) \\ 40 \times 3 &= 10(3 + x) \\ 120 &= 30 + 10x \\ x &= 9 \end{aligned}$$

1. A druggist has a 40% solution of argyrol. How much water should he add to 10 oz. of it to make a 5% solution?

2. How many quarts of water are needed to reduce 4 qts. of an 8% solution of boric acid to a 3% solution?

3. The gold used for coins contains 10% alloy whereas that used for jewelry contains 40%. How much alloy should be added to 20 oz. of coin gold to reduce it to jewelry gold?

4. Pewter is made of tin with 15% of lead. How many pounds of tin must be added to 12 lbs. of pewter to reduce the lead to 10%?

Honor Work

5. A certain iron contains 5% chromium. How much chromium must be added to 850 lbs. of it to make a stainless steel that contains 15% chromium?

6. How many quarts of pure water must be added to 1 qt. of a 20% solution of ammonia to reduce it to a 5% solution?

7. A certain solution contains 20% sugar. How much water must be evaporated from 30 gal. of it to produce a 66⅔% sugar solution?

8. Milk contains 4% butter fat. If a city requires milk dealers to deliver a milk containing at least 3% butter fat, how much cream containing 30% butter fat may a milk dealer remove from 1080 qts. of milk?

9. A druggist has two solutions of iodine, one containing 5% and the other 20%. How much of each should he use to make 8 oz. of a solution to contain 10% iodine?

10. A milkman has milk testing 4% butter fat. If he wishes to make 465 qts. of a special grade-A milk to test 5% butter fat, how much cream testing 35% butter fat must he add to his milk?

11. How much pure water must be added to 3 qts. of a 40% solution of hydrochloric acid to reduce it to a 6% solution of the acid?

The Formula in Medicine and Pharmacy

Honor Work

Modern physicians and druggists often use formulas instead of the methods you have just learned for changing the percentage of a mixture. They reduce n oz. of an $x\%$ solution to a $y\%$ solution by adding $w = n\left(\frac{x}{y} - 1\right)$ oz. of water. More often still the doctor writes "*Aqua q s ad ℥* viii," which means "Add enough water to make 8 oz.," leaving it to the druggist to complete the work.

Illustration 1. Reduce 3 oz. of a 40% solution of ether to a 10% solution.

$$w = n\left(\frac{x}{y} - 1\right) = 3\left(\frac{40}{10} - 1\right) = 9$$

The druggist would add 9 oz. of water.

To make up n c.c. (cubic centimeters) of a $y\%$ solution by reducing an $x\%$ solution, take $s = \frac{ny}{x}$ c.c. of the solution and add $w = \frac{n(x - y)}{x}$ c.c. of water.

Illustration 2. To make 5 c.c. of a 24% solution from an 80% solution, take $s = \frac{5 \times 24}{80} = \frac{3}{2}$ c.c. of the solution and $w = \frac{5(80 - 24)}{80} = \frac{7}{2}$ c.c. of water.

Check: $\frac{3}{2} + \frac{7}{2} = \frac{10}{2} = 5$

$80\% \times \frac{3}{2}$ c.c. $= 1.20$ c.c. $\qquad 24\% \times 5$ c.c. $= 1.20$ c.c.

1. How many cubic centimeters of oil should be added to 100 c.c. of a 50% solution of camphor to make a 20% solution?

2. How many ounces of alcohol should be added to 8 oz. of a 50% solution of chloroform to make a 30% chloroform liniment?

3. How many ounces of a 15% solution of silver nitrate, and how many ounces of water should be used to make 12 oz. of a 2% solution of silver nitrate to be used in the eyes?

4. How much water should be added to a 50% solution of phenol to make 500 c.c. of a 5% solution for disinfecting?

5. Find the amounts of water and of a 20% solution of boric acid needed to make 1000 c.c. of a 5% solution.

Review Problems

1. One number is twice a second number, and the second number is 3 times a third number. Their sum is 70. Find the numbers.

2. A man lends part of $7200 at 4% interest and the remainder at 5%. The income is the same for both loans. How much did he lend at each rate?

3. Albany and New York are 150 mi. apart. A car leaves Albany for New York and averages 40 mi. an hr. Another car leaves New York for Albany at the same time and averages 35 mi. an hr. How far from Albany will they meet?

4. The difference between two numbers is 72 and $\frac{1}{8}$ of the larger equals $\frac{1}{2}$ of the smaller. Find the numbers.

5. A man earns 4 times as much a day as his son. One week the man worked 5 days and the son 6 days. Their combined wages were \$52. How much did each receive a day?

6. John and Henry started from the same place and rode their bicycles in opposite directions. John started at 9 A.M. and rode 8 mi. an hr. Henry started 1 hr. later and rode 10 mi. an hr. How long had Henry traveled when they were 35 mi. apart?

7. Grace paid \$2.30 for stamps. She bought 3 times as many 3¢ stamps as 5¢ stamps and twice as many $1\frac{1}{2}$¢ stamps as 3¢ stamps. How many of each did she buy?

8. Tom had \$2000 invested at 7% and \$5000 at 4%. How much additional money must he invest at 6% to make his total income 5% of his total investment?

9. Paul received 73 in English, 80 in French, and 89 in geometry. What mark must he get in history to make an average of 85?

10. Katherine invested \$500, part at 5% and the remainder at 4%. The interest amounted to \$22 annually. How much did she invest at each rate?

11. A merchant wishes to mix candy worth 55¢ a lb. with candy worth 35¢ a lb. How many pounds of each must he use to make a mixture of 20 lbs. worth 40¢ a lb.?

12. How many ounces of water must Mrs. Richardson add to 8 oz. of a 12% solution of argyrol to reduce it to a 5% solution?

13. To make up 200 lbs. of nuts to sell at 50¢ a lb., how many pounds of 60¢ nuts should be mixed with 35¢ nuts?

14. Our baseball team has won 3 games out of 6 played. How many must it win out of the next 9 games to make its average $66\frac{2}{3}$%?

15. Carl must be back in school in $2\frac{1}{2}$ hrs. How far can he ride his bicycle up hill at 3 mi. an hr., if he can ride back at 12 mi. an hr.?

16. How many tablespoons of water should be added to

9 tablespoons of a 60% solution of soda to reduce it to a 40% solution?

17. Eighteen k. gold contains $\frac{3}{4}$ gold and $\frac{1}{4}$ alloy. How much alloy must be added to 16 oz. of 18 k. gold to reduce it to 14 k. gold which is $\frac{7}{12}$ gold?

18. Two trains start at the same time from cities that are 140 mi. apart and travel toward each other at rates of 15 mi. an hr. and 20 mi. an hr. What distance will the slower train have traveled when they pass?

19. How many pounds of 30¢ coffee must be mixed with 45¢ coffee to make a mixture of 300 lbs. worth 35¢ a lb.?

20. How many pounds of peanuts worth 15¢ a lb. should be mixed with 40 lbs. of walnuts worth 50¢ a lb. to make a mixture worth 30¢ a lb.?

21. A baseball player has a batting average of .200 in 80 times at bat, but is now batting .400. If he keeps up this rate, how many times must he bat to bring his average up to .240?

22. A baseball team won .375 of its first 24 games, but will now meet easier opponents from whom it ought to win .750 of its games. How many games must it still play to bring its record up to .525?

23. One angle of a triangle is 50°. A second angle is 30° larger than the third. Find the number of degrees in each angle.

24. An isosceles triangle has two equal angles. The third angle is 20° less than their sum. Find the number of degrees in each angle.

25. Find the acute angles of a right triangle if one of them is 18° more than the other.

26. The second angle of a triangle is twice the first, and the third angle is 20° more than the first. Find the number of degrees in each angle.

27. One angle of a triangle is 3 times another, and the third equals their sum. Find the number of degrees in each angle of the triangle.

28. Two lines are parallel. One of a pair of alternate interior angles contains 38° more than half the other. How many degrees are there in each?

29. Two lines are parallel. One of the two interior angles on the same side of the transversal contains 40° more than 3 times the other. Find the number of degrees in each.

30. Two lines are parallel. One of a pair of corresponding angles equals $3x - 20$ and the other equals $x + 100$. Find the size of each.

31. Find the size of two supplementary angles if one of them is 30° less than twice the other.

32. How large is an angle if it is 56° larger than its supplement?

33. A merchant bought suits at $12 each. At what price must he sell them to make $33\frac{1}{3}\%$ of the selling price?

34. Mr. Haddon invests $12,000, part at 5% and part at 6%. He receives $60 a yr. more from that at 6% than from that at 5%. How much has he at each rate?

35. In a park, 300 benches around the grand stand have a seating capacity of 1700 people. If some of them seat 6 people each and the rest seat 5 each, how many of each kind are there?

36. Find two numbers whose sum is 38 and whose difference is 4.

37. The numerator of a fraction is 3 less than the denominator, and the fraction can be reduced to $\frac{5}{6}$. What is the fraction?

38. The denominator of a fraction is twice the numerator, and if I add 8 to the numerator the fraction becomes 1. What is the original fraction?

39. In building a through road, the state agrees to pay twice as much as the town, and the Federal government will pay as much as both together. How much must a town pay if the road through it costs $21,000?

40. Emily bought a coat at a $\frac{1}{3}$ off sale for $16. What was its former price?

41. A profit of 20% of the selling price is what per cent of the cost?

Review Exercises

1. Add:

$$\begin{array}{r} 6a^2 \\ \underline{8a^2} \end{array} \qquad \begin{array}{r} 4bc \\ \underline{-7bc} \end{array} \qquad \begin{array}{r} -6d^3 \\ \underline{10d^3} \end{array} \qquad \begin{array}{r} -3x^2 \\ \underline{-5x^2} \end{array} \qquad \begin{array}{r} 2m \\ \underline{-2m} \end{array}$$

2. Subtract:

$$\begin{array}{r}15\,x\\9\,x\\\hline\end{array}\qquad\begin{array}{r}8\,y\\11\,y\\\hline\end{array}\qquad\begin{array}{r}4\,c\\-7\,c\\\hline\end{array}\qquad\begin{array}{r}-6\,a\\-6\,a\\\hline\end{array}\qquad\begin{array}{r}-5\,b^2\\-8\,b^2\\\hline\end{array}$$

3. Multiply:

$(5\,x^5)(2\,x^2)$ $\qquad(-7\,ab)(-3\,b^2)$

$(-2\,b)(8\,c)$ $\qquad(x^3)(x)(x^4)$

$(3\,x^3)(-4\,xy)$ $\qquad(-5)(-2)(-3)$

4. Divide:

$$\frac{6\,x^6}{3\,x^3}\qquad\frac{15\,m^3n^2}{-3\,m^2n^2}\qquad\frac{8\,c^3}{8\,c^3}$$

$$\frac{-8\,a^2}{4\,a}\qquad\frac{-20\,b^7}{-4\,b^2}\qquad\frac{5\,ab^3c^2}{2\,a^2b^2c^2}$$

5. Remove parentheses, and combine like terms:

$5\,a+(6\,b-2)$ $\qquad m-(-2\,m-3)$

$3\,c-(5\,d+4)$ $\qquad 3\,k-2(k-6)$

$6\,a-(3\,a-7)$ $\qquad x-2(4-x)-3$

6. Solve and check:

$x+5=8$ $\qquad 3\,y-(y-4)=14$

$x-3=9$ $\qquad 2(9-3\,z)+3(4-z)=21$

$3\,x=12$ $\qquad \frac{k}{2}-\frac{k}{5}=21$

$2\,x-7=17$

7. Find the value of:

$s=\dfrac{a+b+c}{2}$ if $a=17$, $b=12$, and $c=13$

$a=\dfrac{(n-2)180}{n}$ if $n=12$

$R=\dfrac{3\,pm}{2\,wh^2}$ if $p=8000$, $m=200$, $w=6$, and $h=8$

8. Find the value of:

$38\times51-28\times51$ $\qquad 489\times372-479\times372$

$8432\times3769+8432\times6231$

9. Does y increase or decrease as x increases in:

$y=3\,x+4$? $\qquad y=\frac{6}{x}$? $\qquad 2\,x+y=20$? $\qquad y=-\frac{12}{x}$?

$y=10-x$? $\qquad xy=18$?

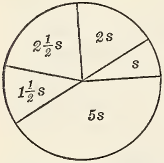
10. If 3 apples cost x cents, what does 1 apple cost?

How many hours will it take to go m mi. at x mi. an hr.?

If 5 books cost d dollars, what will 7 books cost?

A and B were m mi. apart but have traveled toward each other for h hrs. at x and y mi. an hr. respectively. How far apart are they now?

11. Charles made this graph to show how his father's estate was divided among 4 children and their mother, the mother receiving the largest share. If the estate was worth $54,000, find the share of each.



12. The telephone company charges a business house $12 a mo. for a telephone exchange and 30¢ for each phone. Make a formula for the cost of f phones. Find c when $f = 20$, when $f = 40$. When the number of phones is doubled, is the cost doubled?

Test in Making Equations from Problems

Write equations for these problems:

1. If 3 times a number is increased by 12, the result is 30.

2. Six times a number exceeds 42 as much as 22 exceeds twice the number.

3. One number is 7 more than 4 times another number. Twice the larger number exceeds 8 times the smaller number by 14.

4. The second angle of a triangle is 10° more than twice the first angle, and the third angle is 10° more than the sum of the first two.

5. Mr. Washburn sold merchandise for $9 making a profit of 80% of the cost. Find the cost.

6. Two lines are parallel. If one of a pair of alternate interior angles were decreased 25° and the other were increased 15°, the larger angle would lack 5° of being twice the smaller.

7. The B and Q Company have tea worth 60¢ a lb. and tea worth 30¢ a lb. How many pounds of each should they use to make 180 lbs. worth 40¢ a lb.?

8. Mr. Arnold has $8000 invested, part at 5% and part at 6%. The part at 5% produces $70 a yr. more income than that at 6%. How much has he at each rate?

9. A sets out on a trip at 20 mi. an hr. Three hours later B follows from the same place at 35 mi. an hr. In how many hours will B overtake A?

10. Dr. Martin has 15 oz. of a solution containing 20% iodine. How much alcohol should he add to reduce it to a 6% iodine solution?

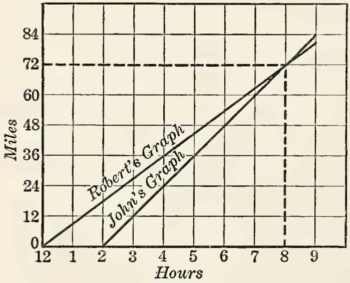
Chapter 9

SETS OF EQUATIONS

THE GRAPH IN MOTION PROBLEMS

MANY problems can be solved graphically without making an equation. Then from the graph, many other questions can be answered that could not be answered from the equation. Whereas one set of equations solve a single problem, the graph solves a group of related problems.

Illustration 1. Robert started for the country on his bicycle at noon, traveling 9 mi. an hr. John will get out of school at 2 P.M. and can ride 12 mi. an hr. He wants to know how quickly he can overtake Robert and how far they will then have gone. John made this graph. Since Robert goes 9 mi. an hr. and started at noon, how far will he have traveled at 1 P.M.? At 2 P.M.? At 3 P.M.? If John starts at 2 P.M. and travels 12 mi. an hr., how far will he have gone at 3 P.M.? At 4 P.M.? At 5 P.M.?



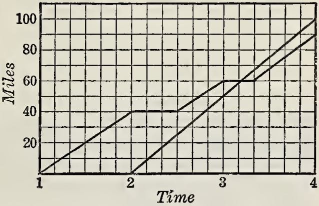
Can you tell from John's graph at what time he will overtake Robert and how far they will then have gone?

When were they 8 mi. apart?

If Robert had stopped traveling at 4 P.M., at what time would John have overtaken him?

How far apart were they at 7 P.M.? If they continue riding, how far ahead will John be at 9 P.M.?

Illustration 2. A local train leaves Chicago for the west at 1 P.M. traveling 40 mi. an hr. It stops from 2 P.M. until 2:30 P.M. at a station, then travels at the same speed until 3 P.M. when it makes a 20-min. stop. It then goes on at the same rate. A through express leaves Chicago for the west at 2 P.M. averaging 50 mi. an hr. At what time and how many miles from Chicago should the train dispatcher arrange to have the express pass the local?



What does the horizontal part of the graph mean? At what time does the express overtake the local? How far from Chicago are they then? How far apart are they at 2:30 P.M.? How far does the express travel while the local is standing at the first station? At what time are they 30 mi. apart?

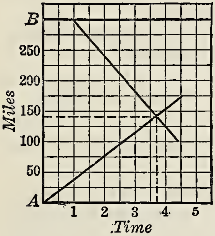


Illustration 3. A and B are 300 mi. apart. A train leaves A at noon for B averaging 40 mi. an hr. At 1 P.M. a train leaves B for A going 60 mi. an hr. At what time and how far from A will they pass?

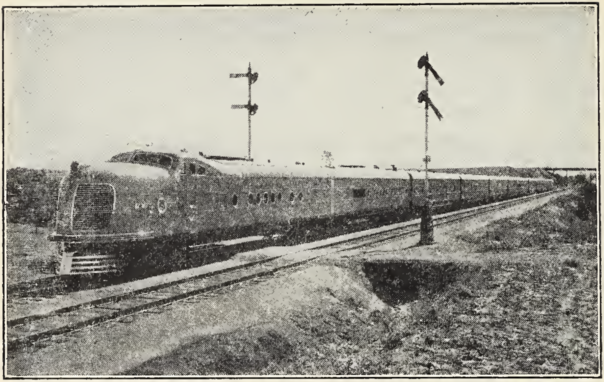
How far apart are the trains at 1 P.M.? At 2 P.M.? At 4 P.M.? At what time are they 100 mi. apart?

By extending the graph, find at what time the train from B will arrive at A. How far from A will the train that left A then be?

Exercises

Solve these problems by graphical methods:

1. At noon a train leaves Kansas City for St. Louis at the rate of 40 mi. an hr. At 1 P.M. it is forced to slow down to 25 mi. an hr. At 2 P.M. another train follows on the same track at 50 mi. an hr. At what time and how far from Kansas City must the train dispatcher arrange to have them pass?



Courtesy of Union Pacific Railroad.

A MODERN STREAMLINED TRAIN

Much mathematical research plays a part in reducing the wind resistance of trains, airplanes, and automobiles of today.

2. A patrol boat that can travel 35 mi. an hr. sights a smuggler 15 mi. away and traveling away from it at 28 mi. an hr. How far must the patrol boat go to overtake the smuggler?

3. A train leaves New York for Philadelphia traveling 45 mi. an hr. At the same time another train leaves Philadelphia, 90 mi. away, and travels toward New York at 30 mi. an hr. In how many hours will they meet and how many miles from New York?

4. An automobile leaves town at 25 mi. an hr. Three hrs. later a second automobile pursues it. At what rate must it travel to overtake the first in 5 hrs.?

5. At noon a car leaves town going 20 mi. an hr. At what rate must a second car travel if it leaves the same place at 1 P.M. and plans to overtake the first when it has gone 70 mi.?

6. Dorothy has $5 and saves $1 a week. If Robert saves $1.50 a week, in how many weeks will he have as much money as Dorothy? How much more money has Dorothy than Robert at the end of 5 weeks?

7. Paul sets out at 9 A.M. by bus going 10 mi. an hr. At 11 A.M. John follows by car at 18 mi. an hr. How far apart are they at noon? At what time will John overtake Paul? At what time will John be 12 mi. ahead of Paul?

8. It costs a publisher $2000 to prepare a book and make plates and $600 extra for every 1000 books sold. If the book sells for $1 a copy, how many thousand copies must be sold before they begin to make money? How much will they make if they sell 10,000 copies?

9. A and B start toward each other at 8 A.M. from two towns 240 mi. apart. A travels 15 mi. an hr. and rests from 12 noon until 2 P.M. on the way, while B travels 25 mi. an hr. and rests from 11 A.M. until 2 P.M. At what time will they meet?

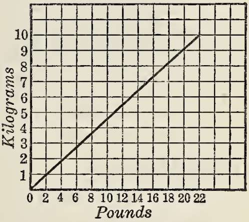
10. A starts from a town at noon and travels east 30 mi. an hr. If B starts at 2 P.M. from another town 20 mi. farther west, at what speed must he travel to overtake A at 6 P.M.?

11. Two men, 60 mi. apart, travel toward each other. One travels 4 mi. an hr. The other starts 1 hr. later and travels 3 mi. an hr. In how many hours will they meet and how far will each have traveled?

The graph as a ready reckoner.

12. This graph shows the relation between pounds and kilograms. For example, 5 kilograms (Kg.) equal 11 lbs. Can you find this on the graph?

How many pounds equal 3 Kg.? 8 Kg.? 10 Kg.? How many kilograms equal 4 lbs.? 9 lbs.? 15 lbs.?

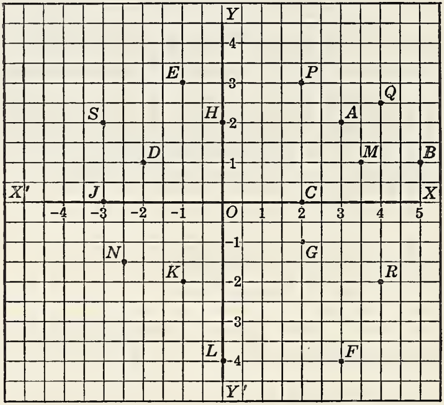


HISTORICAL NOTE ON SETS OF EQUATIONS: The use of two unknowns seems to have been discovered by the Chinese long before the time of Christ but they had no suitable symbolism for them. Later the Hindus used colors, such as red and blue, to stand for two unknowns. About the sixteenth century the letters A, B, and C began to be used. These equations were then solved by the addition-subtraction method.

GRAPHS OF FUNCTIONS

How to map the plane. You have already made graphs of formulas in which you found points that corresponded to two numbers. This same method may be used for locating any point of the plane.

We shall call the horizontal axis, XX', the x-axis; the vertical axis, YY', the y-axis; and the point where they cross, the ***origin.***



Where is 0 placed? Is 0 the same on both axes?

Does positive x run to the right or to the left? Negative x?

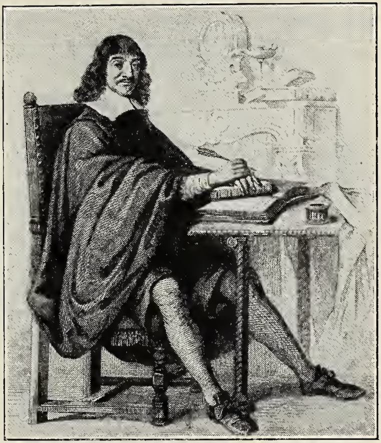
Which way does positive y run? Negative y?

Can you find a point that is 2 in the x direction and also 3 in the y direction? Charles says that this point is P on the graph. Is he right?

Can you find a point that is -3 in the x direction and $+2$ in the y direction? Sarah says that S is the required point because it is 3 units to the left and 2 units up.

The x distance of a point is called the ***abscissa,*** and the y distance is called the ***ordinate*** of a point. Both together are the ***coördinates*** of the point.

Here is a short way to record the coördinates of a point. Write the x value first, then a comma, then the y value, and finally enclose all in a parenthesis. For example P is (2, 3), Q is (4, $2\frac{1}{2}$), R is (4, – 2), and S is (– 3, 2).



RENE DESCARTES (1596–1650)

Famous French philosopher and mathematician. By plotting the graph of algebraic functions, he established a connection between algebra and geometry. In addition to his work in mathematics, he is considered one of the founders of modern philosophy.

Exercises

Make a graph like that shown here, and locate the following points on it:

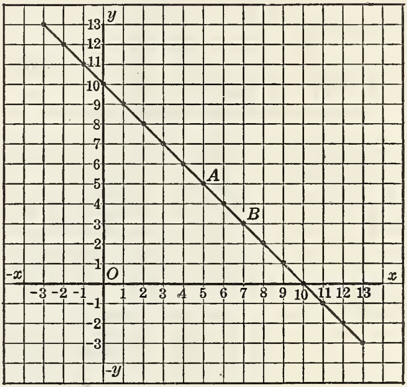
1. (1, 2)
2. (4, 7)
3. (5, – 1)
4. (3, – 2)
5. (– 4, 3)
6. (– 2, 2)
7. (3, – 6)
8. (– 4, – 7)
9. (– 5, – 1)
10. ($2\frac{1}{2}$, 3)
11. ($3\frac{1}{2}$, – 4)
12. (– 1, – $4\frac{1}{2}$)
13. (4, 0)
14. (0, – 3)
15. (– 2, 0)
16. (0, 0)
17. Write the coördinates of points A, B, C, . . . N.

Thought Question

In what direction would point P move if its x value grew larger but its y value remained unchanged?

How to make the graph of an equation. Find two numbers whose sum is 10. If x and y are the numbers, the equation is $x + y = 10$.

Gladys says that $x = 5$ and $y = 5$. She put point *A* on the graph. Is she right?



Katherine says that $x = 7$ and $y = 3$. She put point *B* on the graph. Is she right?

Is there any one solution to this problem?

Can you find other solutions to this equation and put other points on the graph? Can one of the letters be negative?

The class made this table of values for x and y:

x	1	2	3	4	5	6	7	8	9	10	11	12	− 1	− 2	− 3
y	9	8	7	6	5	4	3	2	1	0	− 1	− 2	11	12	13

Are there other points that satisfy this equation between these points? For instance, could $x = 3\frac{1}{2}$? $5\frac{2}{3}$?

Then the class plotted this graph. They called it the graph of the equation $x + y = 10$.

What do you notice about the location of these points?

Could we have made the graph without plotting so many points? Does every point on that line satisfy the equation? Are there any points not on the line that satisfy the equation?

Exercises

Draw the graphs of these equations:

1. $x + y = 6$
2. $x - y = 2$
3. $3x - y = -3$
4. $y = x + 4$
5. $y = 2x - 3$
6. $y = 2x - 1$
7. $y = 2x + 4$
8. $y = 3x + 4$
9. $2x + 3y = 9$
10. $3x - 2y = 4$
11. $y = \frac{1}{2}x$
12. $x + y = 0$

How to solve equations by graphs.

Illustration. Solve the equations:

$$\begin{aligned} 3x + 2y &= 12 \\ 3x - y &= 3 \end{aligned}$$

First, let us make a table of values for each equation. Since two points are needed for locating a line and one other point for a check on our work, we shall make a table of three sets of values for x and y. Of course we shall choose the easiest points to find, those for which one letter is 0, as this will save us the trouble of transposing. So we let $x = 0$ and find the value of y, then let $y = 0$ and find x. Then any other value of x will do for the checking point.

Table for:
$3x + 2y = 12$

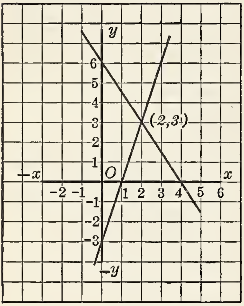
x	0	4	1
y	6	0	$4\frac{1}{2}$

Table for:
$3x - y = 3$

x	0	1	3
y	-3	0	6

Making the graphs of both equations on the same axes, we have the figure on page 196.

What are the coördinates of the point where the lines cross?



Does every point on the graph of $3x + 2y = 12$ satisfy the equation? Does (2, 3) satisfy the equation $3x + 2y = 12$?

Does every point on the graph of $3x - y = 3$ satisfy that equation? Does (2, 3) satisfy it?

Is there any other point that satisfies both equations?

There is only one solution to the pair of equations—the values of x and y at the point where the graphs cross.

Exercises

Solve graphically:

1. $x + y = 5$
$x - y = 1$

2. $2x + y = 7$
$x - 3y = 0$

3. $x - 2y = 2$
$x + 2y = 6$

4. $2x - 3y = 7$
$x + y = 1$

5. $x - 5y = 2$
$2x - y = -5$

6. $5x + y = 3$
$3x - 2y = -6$

7. $x - y = 0$
$2x + y = 15$

8. $y = 3x - 7$
$y = -x + 5$

9. $x + y = 0$
$x - y = 0$

10. $3x - 5y = 2$
$2x + 3y = 14$

11. $3x + 4y = -4$
$2x - 5y = -18$

12. $y = x + 2$
$y = 3x + 2$

13. $x + y = 5$
$3x + y = 11$

14. $x - y = 5$
$x - 4y = -5$

15. $x - y = -4$
$3x - y = -12$

16. $7x + 5y = 2$
$x + 2y = -1$

17. $x - y = 0$
$2x - 3y = 6$

18. $6x - 7y = 1$
$7x - 8y = 2$

Try to solve these equations graphically:

19. $2x - y = 5$
$4x - 2y = 3$

20. $2x + 4y = 10$
$x + 2y = 5$

21. What did you discover about the graphs of Exercise 19? Do they intersect? Can we find the answers from the graph?

Equations that have no common solution are called ***inconsistent equations.***

The graphs of inconsistent equations are parallel lines, because two lines that have no common point are parallel.

22. What did you discover about the graphs of Exercise 20? Does more than one point satisfy both equations? Many? Is it true that any point that satisfies one equation will satisfy the other? How many answers then have these equations?

Two equations are ***dependent*** if every point on the graph of either is also on the graph of the other. You can change one equation into the other by multiplying every term by some number.

We shall learn more about inconsistent and dependent equations later.

ALGEBRAIC SOLUTION: THE ADDITION-SUBTRACTION METHOD

In solving equations by the graphical method, it is necessary to estimate the roots, and often, as when the lines are nearly parallel, it is difficult to get an accurate result. We shall now learn a method that is generally simpler and more accurate.

Illustration 1. Solve:

$$\begin{aligned} 2x - y &= 4 \\ x + y &= 5 \end{aligned}$$

Do you remember how to solve an equation when there is only one letter in it? If you could get rid of one of the letters here, could you find the value of the other?

Look at these two equations. Is there anything that you can do to them that will get rid of y?

If you knew the value of x, could you find the value of y?

Solution: We can get rid of y by adding the two equations:

$$\begin{array}{rcl} 2x - y &=& 4 \\ x + y &=& 5 \\ \hline 3x &=& 9 \\ x &=& 3 \end{array}$$

Now we can find the value of y by substituting 3 for x in either equation. The second is the easier.

$$\begin{array}{rcl} 3 + y &=& 5 \\ y &=& 2 \end{array}$$

Check your answers by substituting in **both** of the **original** equations.

$$\begin{array}{rcl} 2x - y &=& 4 \\ 2 \cdot 3 - 2 &|& 4 \\ 4 &=& 4 \end{array} \qquad \begin{array}{rcl} x + y &=& 5 \\ 3 + 2 &|& 5 \\ 5 &=& 5 \end{array}$$

Illustration 2. Solve: $3x + 2y = 4$
$2x + 3y = 1$

Here we cannot get rid of either letter by adding or subtracting, because the coefficients of neither x nor y are the same in both equations. Is there anything that we can do to make them the same? Can we multiply all terms of an equation by anything we please? What is the smallest number of y that $2y$ and $3y$ will each go into? By what must you multiply the first equation to obtain $6y$? The second equation?

Solution. If we multiply the first equation by 3 and the second by 2, we shall have $6y$ in both equations.

$$\begin{array}{lrcl} \text{Multiplying by 3:} & 9x + 6y &=& 12 \\ \text{Multiplying by 2:} & 4x + 6y &=& 2 \\ \cline{2-4} \text{Subtracting:} & 5x &=& 10 \\ & x &=& 2 \end{array}$$

Substituting in the first:

$$\begin{aligned} 3x + 2y &= 4 \\ 6 + 2y &= 4 \\ y &= -1 \end{aligned}$$

Check by substituting in both of the original equations.

Class Exercises

Solve and check:

1. $3x + y = 7$
   $x - y = 1$
2. $x + 2y = 4$
   $3x + 2y = 8$
3. $x + 2y = 9$
   $x - 2y = 3$
4. $5x - 2y = 4$
   $3x + 2y = 12$
5. $2p + 3q = 5$
   $4p - 3q = 1$
6. $2a + 3b = 8$
   $2a - 5b = -8$
7. $3r - 2s = 6$
   $3r - 7s = -9$
8. $x - 5y = 3$
   $x + 2y = -4$
9. $2x - 3y = 1$
   $3x + 2y = 21$
10. $3x - 4y = -6$
    $4x + 2y = 14$
11. $5x + 6y = -1$
    $3x + 2y = 1$
12. $4v + 3w = 24$
    $3v + 2w = 17$
13. $2h + 5k = 1$
    $3h - 4k = -10$
14. $x + 3 = y$
    $2y - 10 = x$
15. $m - n = 0$
    $n = 3m - 4$
16. $3a + 2b - 29 = 0$
    $b = 2a - 3$

Optional Exercises

17. $2x_1 + 3x_2 = 13$
    $3x_1 - 4x_2 = -6$
18. $5w_1 - 7w_2 = 12$
    $3w_1 - 5w_2 = 8$
19. $\frac{1}{2}x + \frac{1}{4}y = 2\frac{3}{4}$
    $\frac{3}{4}x + \frac{1}{2}y = 4\frac{1}{2}$
20. $2\frac{1}{4}x - 1\frac{1}{2}y = 0$
    $1\frac{3}{4}x + 2\frac{1}{2}y = 22$
21. $.3x + .7y = 1$
    $x - .06y = .94$
22. $.4x + 1.2y = 7.6$
    $1.5x - .08y = 5.6$
23. $\frac{x}{2} - \frac{y}{3} = 2$
    $\frac{x}{4} + \frac{y}{3} = 4$
24. $\frac{2x}{5} + \frac{3y}{8} = 7$
    $\frac{3x}{4} + \frac{2y}{3} = 12\frac{5}{6}$

25. $\frac{x}{2} - \frac{y}{3} = 0$
$y = x - 4$

26. $1.2\,x - .04\,y = 2.4$
$3\,x + 1.5\,y = 6$

27. $10\,\pi - 4\,e = 20.5428$
$3\,\pi + 10\,e = 36.6078$

Honor Work

28. $\frac{2\,x - 1}{3} + \frac{y - 1}{4} = 2$
$\frac{x + 2}{2} - \frac{2\,y - 7}{3} = 1$

29. $\frac{2\,x + y}{2} - \frac{3\,x - 4\,y}{4} = +4$
$2(x + y) - 3(3\,x - y) = -18$

30. $3(x - 1) + 2(2\,y + 1) = -3$
$\frac{x + 5}{3} + \frac{y + 3}{4} = 2$

31. $3\,x - 4\,y = -a + 7\,b$
$2\,x + 3\,y = 5\,a - b$

32. $ax - by = ab$
$bx + ay = 2\,b^2 + a^2$

33. $ax + by = c$
$dx + ey = f$

34. $mx - ny = m^2 + n^2$
$nx + my = m^2 + n^2$

35. $ax + by = a$
$bx - ay = b$

SUBSTITUTION METHOD

Another method of solving sets of two equations is called the substitution method. We first solve one equation for a letter. Then we substitute the value found in the other equation.

Illustration 1. Solve:
$$2\,x + 3\,y = 21 \qquad (1)$$
$$4\,x + y = 17 \qquad (2)$$

Solving (2) for y:
$$y = 17 - 4\,x \qquad (3)$$

Substituting this value of y in (1):
$$\begin{aligned} 2\,x + 3(17 - 4\,x) &= 21 \\ 2\,x + 51 - 12\,x &= 21 \\ -10\,x &= -30 \\ x &= 3 \end{aligned}$$

Substituting $x = 3$ in (3):
$$\begin{aligned} y &= 17 - 12 \\ y &= 5 \end{aligned}$$

Check:
$$\begin{array}{r|r} 2\,x + 3\,y = 21 & 4\,x + y = 17 \\ 6 + 15 \mid 21 & 12 + 5 \mid 17 \\ 21 = 21 & 17 = 17 \end{array}$$

Illustration 2. Solve:
$$3\,a - 5\,b = 22 \qquad (1)$$
$$5\,a + 3\,b = 14 \qquad (2)$$

Solving (1) for a:

$$3a = 22 + 5b$$

$$a = \frac{22 + 5b}{3} \qquad (3)$$

Substituting (3) in (2):

$$5\left(\frac{22 + 5b}{3}\right) + 3b = 14$$

$$110 + 25b + 9b = 42$$

$$34b = -68$$

$$b = -2$$

Substituting $b = -2$ in (3)

$$a = \frac{22 - 10}{3}$$

$$a = 4$$

Check:

$$\begin{array}{r|l} 3a - 5b = 22 & 5a + 3b = 14 \\ 12 + 10 \mid 22 & 20 - 6 \mid 14 \\ 22 = 22 & 14 = 14 \end{array}$$

Exercises

Solve and check:

1. $4x + y = 13$
$5x + 3y = 18$

2. $2x + 3y = 8$
$x - 6y = -11$

3. $2x + y = 8$
$5x - 2y = 11$

4. $x - 2y = 2$
$4x + 3y = -3$

5. $3x - y = 2$
$2x + 3y = 5$

6. $x = 3y + 5$
$y = 2x - 20$

7. $3x - 4y = 2$
$5x + 2y = 25$

8. $2x + 3y = -12$
$5x - 7y = -1$

9. $2x - 5y = .5$
$3x - 7y = 1$

10. $1.5x + 2.8y = 17$
$.5x - y = -4$

11. $2.4x - 3y = 7.8$
$.85x + .01y = 1.69$

12. $3.25x + .45y = 4.15$
$2.75x - y = .75$

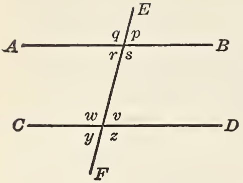
13. In the formula $D = dq + r$, find d and r if $D = 103$ when $q = 8$, and $D = 115$ when $q = 9$.

14. A merchant, using the formula $C = S + L$, finds his loss, L, to be \$8. If the selling price, S, had been 50% greater, L would have been \$2. Find the cost and selling price.

15. Find a and b if the graph of $ax + by = 10$ passes through the points $(1, -2)$ and $(4, 2)$.

16. In the formula $d = vt + \frac{1}{2}at^2$, $d = 104$ when $t = 2$, and $d = 336$ when $t = 4$. Find v and a.

Sets of Equations in Geometry



Is AB parallel to CD if:

1. $r + v = 112°$
$3r - 2v = 56°$
2. $r + v = 210°$
$r - v = 0°$
3. $r + v = 100°$
$5r - 3v = 180°$

Algebraic Expression

If x *is the larger of two numbers and* y *the smaller, represent:*

1. The sum of two numbers is 17.
2. The difference of two numbers is 12.
3. One number exceeds the other by 20.
4. Twice the larger number exceeds 3 times the smaller by 15.
5. The sum of two numbers is 5 times their difference.
6. The quotient of two numbers is 3.

Problems involving equations with two letters. It is often much easier to write the equations for a problem by using two letters than it is by using only one letter. But it is necessary to form as many equations as there are letters. When you use only one letter, you will need only one equation, but when you use two letters, you must make two equations.

Illustration. The sum of two numbers is 15, and twice the larger exceeds the smaller by 9. Find the numbers.

Let x = the larger
y = the smaller
$x + y = 15$
$2x - y = 9$
Solving: $x = 8$
$y = 7$

Check:
The sum of 8 and 7 is 15.
Twice 8, or 16, exceeds 7 by 9.

Algebraic Shorthand

Express the following problems as equations in two letters. Then solve your equations.

1. The sum of two numbers is 21, and the difference of these same numbers is 11.

2. Find two numbers whose quotient is 3 and whose sum is 24.

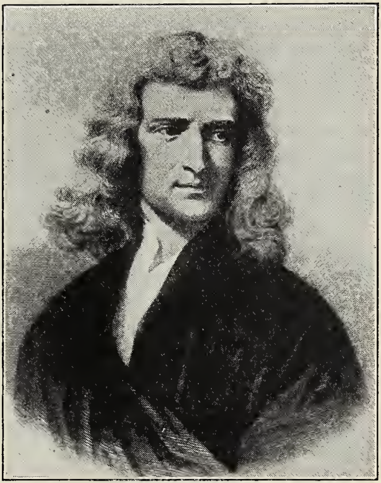
3. Find two numbers whose sum is 11, if twice the first exceeds 5 times the second by 1.

4. Find two numbers whose difference is 6 and whose average is 7.

5. Find two numbers whose average is 8, if the first is 4 more than 3 times the second.

6. The larger of two numbers is 3 more than twice the smaller, and twice the larger is 1 more than 5 times the smaller.

7. If twice a number is added to 3 times a smaller number, the sum is 17. But if twice the smaller is added to 3 times the larger, the sum is 18.



ISAAC NEWTON

Isaac Newton (1642–1727) was the first to solve equations by the substitution method. He discovered the laws of gravity, of moving bodies, and of the propagation of light. His work laid the foundation for modern mechanics and astronomy.

8. Sam Ling charged me 66¢ for laundering 3 shirts and 10 collars one week. The next week he charged me 45¢ for laundering 2 shirts and 7 collars. What is his price for each article?

9. John said to Robert, "I have 6 marbles more than you have." Robert replied, "Tomorrow I shall bring twice as many as I have today, and then I shall have 8 more than you have." How many marbles has each boy today?

10. At a special sale the King Grocery Company gave 3 lbs. of coffee and 5 lbs. of sugar for $1, or 2 lbs. of coffee and 3 lbs. of sugar for 65¢. For what price a pound were they selling each?

11. Mr. Farrell saves $4 a day when he works, but it costs him $3 a day to live when he is idle. If he must save $198 during the next 60 days, how many days must he work?

12. For a football game the tickets cost 25¢ if bought at the school, but 50¢ if bought at the field. The manager says that $270 was taken in for the 960 tickets sold. How many were sold at the school?

13. A club allows its secretary $1 for sending notices of a meeting. If he must notify 58 persons, to how many may he send 2¢ letters and to how many must he send 1¢ post-cards?

14. A contractor says that he will furnish me with 5 men and 3 boys for $44 a day, or with 4 men and 5 boys for $43 a day. What rate is he charging for a man and for a boy?

15. In sending a telegram a certain charge is made for the first 10 words and a fixed amount for each additional word. Dorothy paid 61¢ for a 22-word message and 94¢ for a 33-word message. What are the two rates?

16. Two investments total $9000. Find their amounts if one is at 6% and the other at 5%, and the total income per year is $504.

17. Three times one number added to 5 times a second number gives 49. But if I subtract 3 times the second from twice the first, the remainder is 1. Find the numbers.

AREA PROBLEMS

Many area problems can be expressed easily in terms of a single letter, but there are many others that either require two letters or are much more easily stated in terms of two letters.

Area Problems Requiring but One Letter

1. The width of a rectangle is w, and its length exceeds its width by 5. What is its length? Its perimeter? Its area?

2. The width of a rectangle is w, and its area is $w^2 + 4w$. Write an expression for its length.

3. The length of a rectangle is 7 more than the side of a square, and the width is 3 less than the side of the square. Letting s represent the side of the square:

(*a*) Represent the dimensions of the rectangle.

(*b*) Represent the area of the rectangle.

(*c*) Represent the perimeter of the rectangle.

(*d*) State algebraically that the area of the rectangle is 200.

(*e*) State that the length is twice the width.

(*f*) State that the area of the rectangle exceeds that of the square by 40.

(*g*) State that the sum of the perimeters is 90.

(*h*) If the length of the rectangle were decreased 3 and the width increased 4, express the new dimensions, the new area.

(*i*) State that the area of the new rectangle exceeds that of the original rectangle by 50.

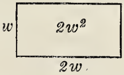
Illustration. The length of a rectangle is twice the width. If the length is increased 3 and the width diminished 1, the area is increased 3. Find the dimensions.

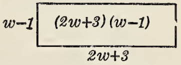
In solving geometry problems, it is best to draw the figures and write the dimensions and areas on them.

Since the area is increased by 3, the second area is 3 more than the first, or $(2w + 3)(w - 1) = 2w^2 + 3$.

$w = 6$, the width

$2w = 12$, the length





4. The length of a rectangle is 10 ft. more than the width. If each dimension is increased 5 ft., the area will be increased 475 sq. ft. Find its dimensions.

5. The length of a rectangle is 3 times the width. If the length is decreased 4 and the width is increased 3, the area will be increased 8. Find its dimensions.

6. Find the side of a square whose area is increased 32 when its dimensions are each increased 2.

7. If the base of a square is increased 3 and the altitude decreased 2, the area remains unchanged. Find its side.

8. Mrs. Hart has the choice of two rugs, both having the same area for her square dining room. One is the length of the room but 4 ft. narrower, whereas the other is 2 ft. shorter than the room and $2\frac{1}{2}$ ft. narrower. Find the dimensions of the room.

Area Problems in Two Letters

1. The width of a rectangle is w and its length is l. Express its perimeter, its area.

2. Using l and w for the length and width of a rectangle, write an equation for:

(*a*) Its perimeter is 34. (*b*) Its area is 90.

(*c*) The length is 3 less than twice the width.

(*d*) Twice the length exceeds 3 times the width by 5.

(*e*) If the length were increased by 3, the area would be 50.

(*f*) If the width were decreased by 8, the area would be decreased by 70.

(*g*) If the length and width are each increased by 4, the area is doubled.

3. If l is the length and w the width of a rectangle, express in words:

(*a*) $l + 3$ and $w - 2$ (*c*) $lw + 64$

(*b*) $(l + 3)(w - 2)$ (*d*) $(l + 3)(w - 2) = lw + 64$

4. If the length of a rectangle is increasing, but its width remains unchanged, what change is taking place in its area? In its perimeter?

5. Twice the length of a rectangle exceeds 3 times the width by 1. If each dimension is decreased by 2, the area is decreased by 22. Find the width.

6. If the length of a rectangle is increasing, but its area remains unchanged, what change is taking place in its width?

7. If the length of a rectangle is increased by 3 and the width increased by 2, the area is increased by 40. But if the length is increased by 2 and the width by 3, the area is increased by 42. Find the original length and width.

8. If the length of a rectangle is increased by 1 and the width decreased by 1, the area is decreased by 3. But if the length is decreased by 1 and the width increased by 2, the area is increased by 9. Find the length and width.

9. If the length of a rectangle is decreased by 3 and the width increased by 2, the area remains unchanged. Also if the length is decreased by 4 and the width increased by 3, the area is unchanged. Find the dimensions.

10. If I cut a strip 1 in. wide from all sides of a sheet of paper, I reduce its area by 40 sq. in. If, instead, I cut a strip the same width from the two ends only, the sheet becomes a square. Find its dimensions.

OUR NUMBER SYSTEM

The number system that we use today was invented by the Hindus and came to us through the Arabs. Its great advantage over all older systems will be evident to you if you attempt to perform any of the ordinary operations with, for instance, the Roman numerals. For example, try to multiply LXXIX by XIV. Even a simple addition is difficult. Try XLII + XXVIII.

The figures 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0 are called **digits.** When two digits are placed together, the number formed is the sum of the right-hand digit and 10 times the left-hand digit. For example, $37 = 10 \times 3 + 7$. When the digits are reversed, a new number is formed, for now it is the other digit that is multiplied by 10. $73 = 10 \times 7 + 3$. If the tens' digit is t and the units' digit is u, the number is $10t + u$.

Questions on Digits

1. How many units are there in a number if:

(*a*) Its tens' digit is 4 and its units' digit is 7?

(*b*) Its tens' digit is x and its units' digit is y?

2. Write a number whose tens' digit is 5 and whose units' digit is n, one whose tens' digit is n and whose units' digit is 5.

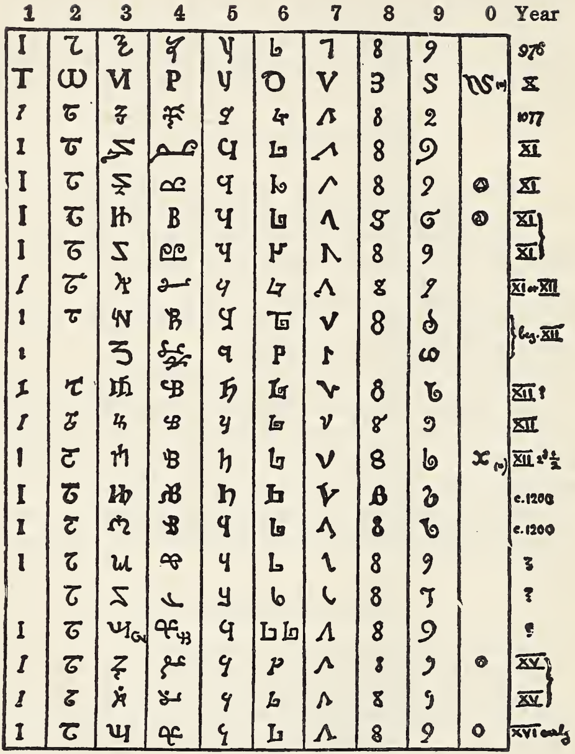
3. If the tens' digit of a two-digit number is t and the units' digit is u:

(*a*) Write the number.

(*b*) Write the number having these digits reversed.

(*c*) Write the sum of the digits.

(*d*) If $t = 6$ and $u = 9$, what is the number? What is the number with the digits reversed? What is the sum of the digits?


1
2
3
4
5
6
7
8
9
0
Year
976
X
1077
XI
XI
XI
XI
XI or XII
beg. XII
XII ?
XII
XII 2nd ½
c.1200
c.1200
XV
XV
XVI early

NUMBER SYSTEMS

Our present number system is the culmination of the experiments of many people for centuries. The Arabs, Persians, Egyptians, and Hindus all probably had a hand in originating our numerals. Some of the numerals are found in inscriptions made in the 3rd century B.C. Numerals are thought first to have appeared in Europe in 976 A.D. From Hill, *The Development of Arabic Numerals in Europe* by permission of Oxford University Press, publishers.

Illustration. The tens' digit of a two-digit number exceeds twice the units' digit by 2. The number is 27 more than the new number formed by reversing the digits. Find the number.

	Tens' Digit	*Units' Digit*	*Number*
Original number	t	u	$10t + u$
Reversed number	u	t	$10u + t$

$$\begin{aligned} t - 2u &= 2 \\ 10t + u &= 27 + 10u + t \\ u &= 1 \\ t &= 4 \\ 10t + u &= 41 \end{aligned}$$

Check:
4 exceeds 2 × 1 by 2.
41 is 27 more than 14.

Exercises on Digits

1. The units' digit of a certain two-digit number is twice the tens' digit. If 18 is added to the number, the digits will be reversed. Find the number.

2. The sum of the digits of a two-digit number is 6, and if its digits are reversed, the new number exceeds the original by 18. Find the number.

3. The sum of the digits of a two-digit number is 7. If the digits are reversed, the new number is 2 more than twice the original number. Find the original number.

4. A two-digit number is 3 times the sum of its digits, but if the digits are reversed, the new number is 45 more than 3 times the sum of its digits. What is the number?

5. A certain two-digit number exceeds the sum of its digits by 18 and exceeds the number formed by reversing the digits by 9. Find the number.

6. If a two-digit number is divided by the sum of its digits, the quotient is 4. If it is divided by 1 more than the sum of its digits, the quotient is 3. Find the number.

7. A two-digit number is 6 times its units' digit. If the digits are reversed, the sum of the new number and the original number is 66. Find the original number.

8. The sum of the digits of a two-digit number is 8. If the digits are reversed, the number is not changed. Find the number.

USING TWO UNKNOWNS

Many other problems of types that you have already solved in this course using only one letter, can be stated more easily in terms of two letters.

Illustration 1. Find two numbers whose sum is 26 and whose difference is 10.

Using x and y:

$$x + y = 26$$
$$x - y = 10$$

Illustration 2. Mr. Head invested \$10,000, partly at 5% and partly at 6%. If his annual income was \$570, how much had he at each rate?

Using x and y:

Principal	*Rate*	*Income*
x	.05	$.05\,x$
y	.06	$.06\,y$
10,000		570

$$x + y = 10000$$
$$.05\,x + .06\,y = 570$$

Illustration 3. Mr. Merchant has some coffee worth 40¢ a lb. and some worth 25¢ a lb. How much of each should he take to make 150 lbs. worth 35¢ a lb.?

Using x and y:

Kind	*No. of Lbs.*	*Value*
40	x	$40\,x$
25	y	$25\,y$
35	150	35(150)

$$x + y = 150$$
$$40\,x + 25\,y = 35(150)$$

If you use initial letters for the unknowns, you can often write your equations as shorthand without filling the boxes.

Problems

Solve these problems using two letters.

1. To make 200 lbs. of tea worth 40¢ a lb., how many pounds of 50¢ and of 35¢ tea must I mix?

2. From an investment of $8000, part at 4% and part at 5%, Mrs. Dyke has an annual income of $350. How much has she at each rate?

3. On a trip of 40 mi. Paul walked part way at 4 mi. an hr. and then got a ride the rest of the way at 12 mi. an hr. He arrived in 6 hrs. How far did he walk, and how far did he ride?

4. Mr. Broker sold 50 bonds for $40,000. Some were $500, and some $1000 bonds. How many of each did he sell?

5. In making up 500 lbs. of mixed nuts to sell for 40¢ a lb. Allan used some nuts worth 50¢ a lb. and some worth 25¢ a lb., How much of each did he use?

6. Emily and Sarah together weigh 200 lbs. On a teeter board they balance when Emily is 6 ft. from the support and Sarah is 4 ft. from it. What are their weights.

7. Jones lends part of $7200 at 4% and the rest at 5%. The incomes from the two parts are equal. How much has he at each rate?

8. For a high-school entertainment 186 tickets were sold. Adults were charged 50¢ and children 25¢. If the receipts amounted to $75, how many tickets of each kind were sold?

9. The difference between two numbers is 32, and $\frac{1}{6}$ of the larger equals $\frac{1}{4}$ of the smaller. What are the numbers?

10. A man's daily wages were 4 times his son's. One week he worked 5 days and his son worked 6 days. Their combined pay was $39. How much did each earn a day?

11. Last week, Mr. Bartley, a salesman, included 6 dinners and 4 lunches in his expense account for which he asked $5.10. This week he charged the company $5.90 for 7 lunches and 5 dinners. What price does he pay for each meal?

12. Smith invested $5000, part at 5% and part at 4%. His yearly income was $220. How much did he invest at each rate?

13. The units' digit of a two-digit number exceeds the tens' digit by 5. The number increased by 63 equals 10 times the sum of its digits. Find the number.

14. If the width of a rectangle were doubled and the length were decreased by 3, the figure would be a square whose area would be 60 less than double the area of the rectangle. What are the dimensions of the rectangle?

Review Problems

1. Mr. Stearns has shares of stock A and of stock B. Last week stock A went up \$3 a share, and stock B went up \$2 a share. Consequently Mr. Stearns made \$230. But this week, although stock A went up \$1 a share, stock B went down \$4 a share, and he lost \$110. How many shares of each stock has he?

2. Robert sat 8 ft. from the fulcrum of a seesaw and balanced John who sat 6 ft. from the fulcrum. Then Robert moved back to a distance of 9 ft. from the fulcrum and held his dog weighing 9 lbs. in his arms, and it was necessary for John to move back to a point $7\frac{1}{2}$ ft. from the fulcrum to balance him. What are their weights?

3. In mathematics Charles got 72, 90, 70, and 80 in 4 tests. What must he get in the next test to make his average 80 for the 5 tests?

4. The train dispatcher arranged to send out a local train at 3 A.M. and an express at 6 A.M. When he had decided on the average rates of these trains, he computed that the express must pass the local at 10 A.M. This was too early so he decided to increase the speed of the local by 6 mi. an hr. and have the express overtake it at 1:30 P.M. What rate had he planned at first for each train?

5. A man invested one sum at 4% and another at 6% which gave him an income of \$900 a year. The next year he was able to invest the first sum at $4\frac{1}{2}\%$ and the second at 7% which increased his income to \$1030 a year. How much money had he in each investment?

6. A grocer mixed 60¢ coffee with 35¢ coffee and computed that the mixture was worth 50¢ a lb. To reduce it to a value of 45¢ a lb., he then added 50 more lbs. of the cheaper coffee. How many pounds of each did he use at first?

7. The sum of the digits of a two-digit number is 10. If the digits are reversed, the number is increased by 36. Find the number.

8. The sum of the digits of a two-digit number is 13. If 27 is subtracted from the number, the digits will be reversed. What is the number?

Solve these problems graphically:

9. A leaves town at 20 mi. an hr. Three hours later B follows at 35 mi. an hr. In how many hours will B overtake A, and how far will they then have gone?

10. An express leaves New Orleans for Atlanta at 9 A.M. traveling 45 mi. an hr. At 11 A.M. the flyer leaves Atlanta for New Orleans and travels 60 mi. an hr. If the distance from Atlanta to New Orleans is 493 mi., how far from Atlanta must the train dispatcher arrange to have them pass?

11. The leading baseball team is now 10 games ahead of the second-place team and is winning just $\frac{1}{2}$ of its games. The second team is winning $\frac{3}{4}$ of its games. After how many more games will the second team overtake the first?

12. Show on graph paper how the point (x, y) moves, if:

(*a*) $x + y = 8$ (*b*) $y = 3x$ (*c*) $x - y = 2$

(*d*) x varies but y always remains 4

(*e*) y varies but x always remains -2

(*f*) x and y both vary, but their sum is always 10

(*g*) x and y both vary, but y is always 3 more than x

Test in Geometry

1. Define:

(*a*) Right angle

(*b*) Supplementary angles

(*c*) Vertical angles

(*d*) A perpendicular

(*e*) Perimeter

(*f*) Alternate interior angles

2. What is the sum of the angles of a triangle?

3. If 2 lines are parallel, what fact is true about their corresponding angles?

4. Write the formula for the area of:

(*a*) A parallelogram

(*b*) A triangle

(*c*) A trapezoid

(*d*) A circle

Test in Solving Equations

Solve:

1. $x + y = 7$
$x - y = 3$

2. $3x + 2y = 11$
$x + 2y = 5$

3. $7x - 4y = 18$
$3x + 2y = 30$

4. $3x + 8y = 1$
$2x - 11y = 17$

5. $.06x + .045y = 90$
$x - y = 100$

6. $5x - 9y = 54$
$3x + 6y = 21$

7. $.3x + 2y = 13$
$2x + 1.2y = 26$

8. $x + y = 7$
$\frac{x}{2} + \frac{y}{3} = 3$

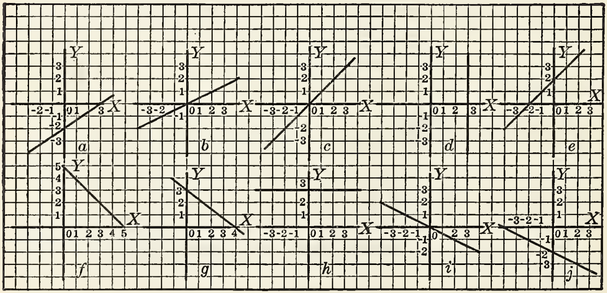
9. $2x - y = 4$
$\frac{x+2}{4} - \frac{y-3}{5} = 1$

10. $2x + y - (x - y) = 22$
$\frac{x+y}{4} - \frac{x-y}{8} = 3\frac{1}{2}$

Matching Test in Graphs

Write the numbers 1 to 10 in a column, and after each write the letter of the corresponding graph.

1. $y = x + 2$
2. $x - 2y = 0$
3. $x + 2y = -4$
4. $x = 3$
5. $x - y = 0$
6. $y = 3$
7. $x + y = 5$
8. $x + 2y = 0$
9. $2x - 3y = 6$
10. $3x + 4y = 12$



Chapter 10

THE FORMULA APPLIED

LITERAL EQUATIONS

MR. Long is an electrical engineer and every day he encounters a large number of problems in which he must use the formula

$$C = \frac{En}{R + nr}$$

to find n. Of course Mr. Long could substitute the values of E, R, r, and C in this formula for each problem and then solve his equation, but Mr. Long is too wise to do all that unnecessary work. He first solves the formula for n, once and for all, and then he has only to substitute to get the answers immediately.

Whenever anyone has to use the same formula many times, he always saves labor by first solving it for the unknown letter.

It will be easier for us, if before we attempt to solve this formula, we review what we have learned about solving equations.

What we have already learned about solving equations. To solve the equation $\frac{x}{2} - 3 = \frac{x}{6} + 4$ means to obtain the answer $x = 21$. That is, we must get x alone on the left side of the equals sign and the number on the right side. Our example has a 2 and a 3 on the left side that should not be there and an $\frac{x}{6}$ on the right side that should not be on that side. Solving an equation then consists principally of getting rid of those quantities that are where we do not want them.

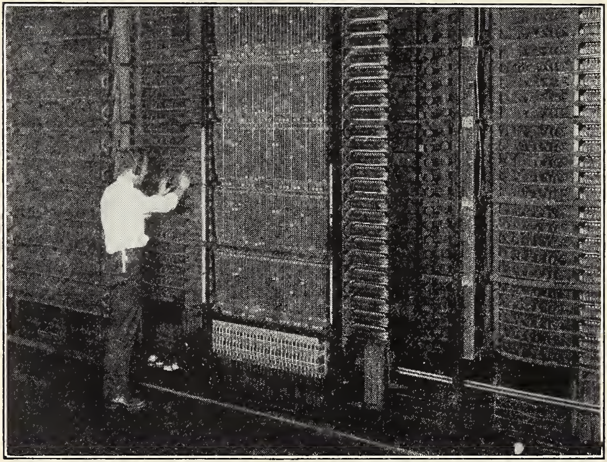
To get rid of a quantity, we perform an operation exactly opposite to that which connects the quantity with the unknown letter.

Addition and subtraction are opposites. If I add 2 to 7 and then subtract 2, I get back the original 7. If I subtract 3 from 5 and then add 3, the answer is again the original 5.

To get rid of a quantity that is added, subtract that amount from both members.

Subtract 4:

$$\begin{array}{rcr} x + 4 & = & 13 \\ 4 & = & 4 \\ \hline x \phantom{{}+4} & = & 9 \end{array}$$



Courtesy of New York Telephone Co.

DIAL TELEPHONE CENTRAL OFFICE

What a mathematics problem this electrical engineer had to solve! Think of designing a machine that can automatically select any one of a million telephones for you as you turn the dial at home!

To get rid of a quantity that is subtracted, add that amount to both members.

Add 7:

$$\begin{array}{r} x - 7 = 6 \\ 7 = 7 \\ \hline x = 13 \end{array}$$

Multiplication and division are opposites. If I multiply 8 by 5 and then divide the result by 5, I get back the 8. If I divide 15 by 3 and then multiply the result by 3, I get back the 15.

To get rid of a multiplier, divide both members by that amount.

$$3x = 18$$

Divide by 3:

$$\frac{3x}{3} = \frac{18}{3}$$

$$x = 6$$

To get rid of a divisor, multiply both members by that amount.

$$\frac{x}{5} = 4$$

Multiply by 5:

$$\frac{5 \cdot x}{5} = 5 \cdot 4$$

$$x = 20$$

Illustration. Solve: $\frac{x}{2} - 3 = \frac{x}{6} + 4$

It is generally best to get rid of denominators first. Since one x is divided by 2 and the other by 6, we must multiply by a number that will contain each of these. 6 will do this.

Multiply by 6: $6 \cdot \frac{x}{2} - 6 \cdot 3 = 6 \cdot \frac{x}{6} + 6 \cdot 4$

$$3x - 18 = x + 24$$

To get rid of 18 from the left side which is subtracted, add 18 to both members.

Add 18:

$$\begin{array}{r} 3x - 18 = x + 24 \\ 18 = 18 \\ \hline 3x = x + 42 \end{array}$$

To get rid of the x on the right side which is added, we must subtract x from both members.

Subtract x:
$$\begin{array}{rcl} 3x & = & x + 42 \\ x & = & x \\ \hline 2x & = & 42 \end{array}$$

We have still to get rid of the 2 that multiplies the x. To do this we must divide both members by 2.

Dividing by 2:
$$\frac{2x}{2} = \frac{42}{2}$$
$$x = 21$$

The literal equation. In a literal equation, some of the letters represent known quantities whose values have not yet been substituted. Except in formulas, where initial letters are used for both known and unknown quantities, it is customary to let the last part of the alphabet represent unknown quantities and the first part known quantities. For example, in $ax + by = c$, a, b, and c are supposed to be known, whereas x and y stand for unknown quantities.

How to solve literal equations. Actually we solve literal equations in exactly the same way that we have solved other equations. But we must be careful to keep in mind which letter is the unknown quantity.

Illustration 1. Solve: $ax + b = c$. Here x is the unknown letter. The b is added. To get rid of it, subtract:

$$ax = c - b$$

The a multiplies the x. To get rid of it, divide by it.

$$x = \frac{c - b}{a}$$

Check:
$$\begin{array}{r|l} \dfrac{a(c-b)}{a} + b & c \\ c - b + b & c \end{array}$$
$$c = c$$

Illustration 2. Solve $mx - n = m - nx$. x is the unknown. We must get all terms containing x in the left member and all terms not containing x in the right member, so n and nx are on the wrong sides. Both are subtracted, so add:

$$mx + nx = m + n$$

Since x occurs in two places, to find what x is multiplied by, factor. $(m + n)\,x = m + n$ **Check:**

Divide by $m + n$ $\quad x = \dfrac{m + n}{m + n} = 1$ $\qquad m(1) - n \mid m - n(1)$

$m - n = m - n$

Class Exercises

Solve for the last letter of the alphabet occurring in the equation, and check your answers:

1. $x + a = b$
2. $5y = 6m + 3y$
3. $mx + n = 2m + n$
4. $hv - k = ha - k$
5. $ay - 5 = 3a - 5$
6. $3x + m = 7m$
7. $ct - 4d = 2d - 6d$
8. $7x - b = 5b + 4x$
9. $mx + k = k + am$
10. $bx + 4c = 4c$

Optional Exercises

11. $ax - 2b = 2a - bx$
12. $ax - ab = a^2$
13. $hv - kv = k - h$
14. $mz - m = md$
15. $8t - 15 = 5a + 3t$
16. $m_1v - 3m_1 = m_1m_2 + 2m_1$

Honor Work

17. $az + a = a + 1$
18. $\pi x + 3k = 4k$
19. $kx - 1 = x - k$
20. $ay + a = -a^2 - y$
21. $ax + 9b^2 = a^2 - 3bx$
22. $ax + b = cx + d$

Using the answer to Exercise 22 as a formula, solve:

23. $7x + 5 = 3x + 1$
24. $hx + k = mx + n$

How to solve the formula for a given letter. Now we are able to solve Mr. Long's formula for n.

$$C = \frac{En}{R + nr}$$

Since one term is divided by $R + nr$, we shall multiply by that quantity first.

$$C(R + nr) = \frac{En(R + nr)}{R + nr}$$

Getting rid of the common factor from numerator and denominator, we have:

$$C(R + nr) = En$$

Multiplied out: $$CR + Cnr = En$$

Now remember that n is the unknown quantity, so get n on the left side only. Also get rid of CR from the left side.

$$\begin{array}{rcl} CR + Cnr \phantom{{}+ En} & = & En \\ CR \phantom{{}+ Cnr} + En & = & En + CR \\ \hline Cnr - En & = & -\ CR \end{array}$$

To find n, we must divide by the number that multiplies it, so first factor out n to get its multiplier together.

$$n(Cr - E) = -\ CR$$

Dividing by $Cr - E$: $$n = \frac{-\ CR}{Cr - E}$$

This is a satisfactory answer, but as Mr. Long never has problems that give a negative answer, it would be more convenient for him to have the CR positive. So he multiplies both numerator and denominator by $-\ 1$. This gives him:

$$n = \frac{CR}{E - Cr}$$

To check, substitute any numbers for E, n, R, and r in the original and find C. Then substitute this in your answer and get back the same value of n you used first.

Class Exercises

Solve each formula for the letter following it:

1. $pv = k$ v

2. $p = kvt$ v

3. $S = 2\,\pi rh$ r

4. $A = \frac{1}{2}\,bh$ b

5. $A = p + prt$ t

6. $C = \frac{5}{9}(F - 32)$ F

7. $s = \frac{1}{2}\,at^2$ a

8. $L = wh + lh$ h

9. $A = \frac{1}{2}\,h(B + b)$ h

10. $w_1l_1 = w_2l_2$ l_2

11. $S = 2\,\pi rh + 2\,\pi r^2$ h

12. $S = \frac{1}{2}\,n(a + l)$ l

Find the value of the unknown letters in:

13. $l = a + (n - 1)d$, if $l = 15$ when $n = 4$, and $l = 27$ when $n = 7$

14. $A = \frac{1}{2}h(B + b)$, if $h = 5$ when $B = 10$, and $h = 8$ when $B = 4$

15. $y = mx + b$, if $y = 5$ when $x = 2$, and $y = 7$ when $x = 6$

16. $ax^2 + by^2 = 33$, if $y = 5$ when $x = 2$, and $y = 1$ when $x = 4$

Optional Exercises

Solve:

17. $s = \dfrac{a + b + c}{2}$ for c

18. $\dfrac{1}{f} = \dfrac{1}{p} + \dfrac{1}{p'}$ for p

19. $C = \dfrac{En}{R + nr}$ for R

20. $S = \dfrac{rl - a}{r - 1}$ for r

21. $F = \dfrac{mv^2}{gr}$ for m

22. $r = \dfrac{gs}{g + s}$ for s

Honor Work

These formulas are used in engineering and in science. Solve each for the letter following it:

23. $P = t(a - d)s$ $\quad d$

24. $L = \dfrac{Mt - g}{t}$ $\quad t$

25. $\dfrac{V_1}{V_2} = \dfrac{P_2}{P_1}$ $\quad P_1$

26. $K = \dfrac{\pi r^2 E}{180}$ $\quad E$

27. $\dfrac{P}{A} = S - \dfrac{Cl}{r}$ $\quad C$

28. $C = w\left(\dfrac{L}{2} - x\right)$ $\quad L$

29. $T = \dfrac{M}{(1 - ck)Ad}$ $\quad c$

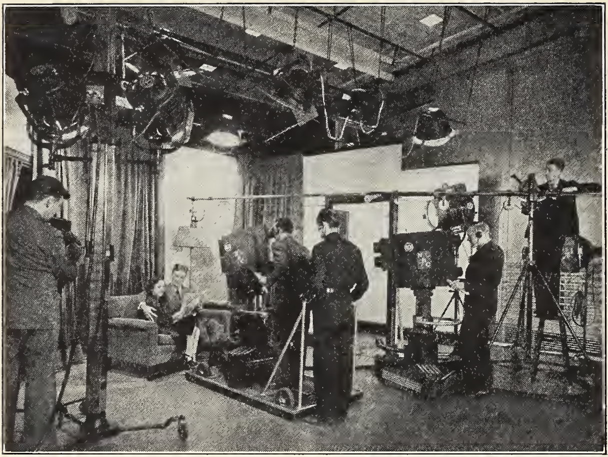
30. $\dfrac{t(a - d)S}{ad} = 1 - \dfrac{d}{a}$ $\quad t$

31. $M = \dfrac{S_1 I_1}{c_1} + \dfrac{S_2 I_2}{c_2}$ $\quad I_2$

32. $v = \dfrac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$ $\quad m_1$

The Formula in Science and Engineering

1. To find two unknown electrical resistances, R and r, an engineer uses the formula $R + nr = \dfrac{2n}{C}$. He finds that when he



Courtesy of Radio Corporation of America.

IN A TELEVISION STUDIO

Designing television apparatus is today one of the most interesting problems for the electrical engineer.

uses 20 cells ($n = 20$), the current (C) is .2; and when he uses 60 cells, the current is .3. What were the unknown resistances?

2. An optician can find the focal length (f) of a lens by measuring the distance (d) from some object to the lens and the distance of the image from the lens (i), and using the formula $\frac{1}{f} = \frac{1}{d} + \frac{1}{i}$. If he wishes to select a lens having a focal length of 10 in., and holds the lens 30 in. from an object, at what distance should he expect the image to appear?

3. Mr. Waters is building a dam and must determine the pressure it will have to stand. He finds that at a depth (d) of 10 ft. the pressure (P) is 58 lbs. and at a depth of 20 ft. it is 101 lbs. If he uses the formula $P = dw + a$, find the value of w and a.

4. A heavy storm has broken down the telephone wires somewhere in the woods. Mr. Burns, the company electrician, knows that the wire has a resistance (R) of 1 for every 100 ft. of wire, so he measures the current through the wire for a voltage (V) of 110, and finds that $C = 2$. Solve the formula $C = \frac{V}{R}$ for R, and from it find how far from the station Mr. Burns should look for the break.

5. In the formula $S = 2\,wl + 2\,hl + 2\,wh$, S is the number of square inches of tin used in making a box whose dimensions are l, w, and h. (*a*) Solve for h. (*b*) John wants to make a box having a base 7 in. by 5 in. What depth should he make the box if his father will allow him to use only 166 sq. in. of tin?

6. The formula $w = \frac{300{,}000}{k}$ is used in radio for changing kilocycles to wave length. (*a*) Solve for k. (*b*) Dorothy finds that the newspaper gives the wave length of KDKA as 306 M. If her radio dial reads in kilocycles, where should she set the pointer to tune in KDKA? (*c*) WGY operates on a wave length of 380 M. How many kilocycles does this represent?

7. The normal weight of a person over 60 in. tall is given by the formula $w = 5\frac{1}{2}(i - 40)$, where i is the height in inches. Find w when i is (*a*) 64 in. (*b*) 68 in. (*c*) 72 in.

THE GENERAL SOLUTION

Many times throughout this book you had to solve several problems of the same kind but containing different numbers. For such groups of exercises you could have made a formula yourself, and then it would have been necessary only to substitute the numbers of each exercise to get the answer.

How to make your own formulas. All you have to do to make a formula is to use letters in place of all numbers in the exercise, and solve. This gives you a general solution in which the numbers of any exercise can be substituted. It is customary to use the first letters of the alphabet, *a*, *b*, *c*, *d*, etc., in place of known numbers. Then the last letters of the alphabet are the unknowns for which you must solve.

Illustration 1. Make a formula for solving exercises like:

$$7x + 2 = 4x + 11$$

Make an equation: $ax + b = cx + d$

Transpose: $ax - cx = d - b$

Factor: $(a - c)x = d - b$

Divide by $a - c$: $x = \dfrac{d - b}{a - c}$

Now to solve any number problem, just substitute back the numbers in your answer.

In $7x + 2 = 4x + 11$, $a = 7$, $b = 2$, $c = 4$, and $d = 11$.

Then: $x = \dfrac{11 - 2}{7 - 4} = 3$

Illustration 2. How shall I invest $20,000, part at 4% and part at 7%, so as to have an income of $1040?

Change to: How shall I invest p dollars, part at a and part at b, so as to have an income of i?

Principal	*Rate*	*Income*
x	a	ax
$p - x$	b	$b(p - x)$

$$\begin{aligned} ax + b(p - x) &= i \\ ax + bp - bx &= i \\ ax - bx &= i - bp \\ (a - b)x &= i - bp \\ x &= \frac{i - bp}{a - b} \end{aligned}$$

In this problem $a = .04$, $b = .07$, $p = 20{,}000$, and $i = 1040$.

Then $x = \dfrac{1040 - .07 \times 20{,}000}{.04 - .07} = \dfrac{-360}{-.03} = 12{,}000$

Illustration 3. Solve for x: $3x + 2y = 13$
$4x - 3y = 6$

First solve for x: $ax + by = c$
$dx + ey = f$

Multiply the first by e: $aex + bey = ce$

Multiply the second by b: $\underline{bdx + bey = bf}$

Subtract: $(ae - bd)x = ce - bf$

Divide by $ae - bd$: $x = \dfrac{ce - bf}{ae - bd}$

In our exercise $a = 3$, $b = 2$, $c = 13$, $d = 4$, $e = -3$, and $f = 6$.

$$x = \frac{13 \times (-3) - 2 \times 6}{3 \times (-3) - 2 \times 4} = \frac{-51}{-17} = +3$$

Exercises

Solve these problems by first making a formula for all problems of the same type, and then substituting the numbers of the problem given here in your formula.

1. At what price a pound must a dealer sell steak that cost him 24¢ a lb., to make $33\frac{1}{3}\%$ of the selling price?

2. Solve $5x - (x - 3) = 19$ for x by first solving the equation $ax - (x - b) = c$.

3. One number is 3 more than another and their sum is 17.

4. Separate 27 into 2 parts so that the larger is twice the smaller.

5. Solve $\frac{x}{5} - \frac{x}{10} = 3$ by first solving $\frac{x}{a} - \frac{x}{b} = c$.

6. The supplement of an angle is 10° more than twice the angle.

7. Find x if one of two alternate interior angles of parallel lines is $3x - 10$ and the other is $x + 70$.

8. Two automobiles, headed in opposite directions, leave a town at the same time, one traveling 30 mi. an hr. and the other 35 mi. an hr. In how many hours will they be 455 mi. apart?

9. A grocer has 60 lbs. of candy worth 80¢ a lb. How much candy worth 30¢ a lb. must he put with it to make a mixture worth 40¢ a lb.?

10. Find the side of a square whose area is increased 63 when its side is increased 3.

The Formula in Business

If you studied business arithmetic, you learned three cases of percentage depending on whether you had to find the base, the rate, or the percentage. In algebra, however, it is not necessary to know those separate cases. One formula

$$br = p$$

which you can easily solve for any letter, can be used for all cases.

The *base* is the number on which the percentage is to be computed.

The *rate* is the per cent or hundredths to be taken.

The *percentage* is the product of the base and rate.

In 6% of 300 is 18, .06 is the rate, 300 is the base, and 18 is the percentage. Note that the rate is .06 and not 6.

Illustration. Mr. Pell has an income of $1800 a yr. He spends $360 a yr. for rent. What per cent of his income is his rent?

Here $b = 1800$, $p = 360$, and r is unknown.

Solving the formula for r:
$$\begin{aligned} r &= \frac{p}{b} \\ &= \frac{360}{1800} \\ &= .20 \text{ or } 20\% \end{aligned}$$

1. Mr. Jones invested $3400 at 4%. What interest does he receive?

2. A house worth $12,000 is insured for $7500. For what per cent of its value is it insured?

3. A manufacturing plant valued at $60,000 depreciates at the rate of 5% a yr. Find the amount of depreciation and the value of the property at the end of 1 yr.

4. Mr. Black receives a 2% discount on all bills because he pays them within 10 days. How much does he save in 1 mo. if his bills amount to $5300 for that time?

5. A bankrupt has $18,000 with which to settle a debt of $45,000. What per cent can he pay on claims?

6. The dividend paid by ABC stock is 5% of its present value. If Henry Brown received a dividend check for $650 from this company, how much is his investment worth?

7. A family could save 10% of their food bill by buying for cash. How much could they save in 1 yr. on an account averaging $60 a mo.?

8. Rapid Railway stock pays a dividend of $4 a share. If I can buy this stock for $60 a share, what per cent will I realize on my investment?

9. What per cent can I save by buying 6 cakes of soap for 25¢ if it sells for 5¢ a cake?

10. Mr. Reed pays a property tax of \$180. If the tax rate is 4%, what is the assessed value of his house?

Simple Interest

The amount A of P dollars invested at rate r simple interest for t yrs. is given by the formula:

$$A = P + Prt$$

1. Solve this formula for t.

2. In how many years will \$120 amount to \$148.80 at 6% simple interest?

3. In how many years will \$200 double itself at 5% simple interest?

4. Solve for r.

5. What rate must I receive on \$1800 so that it will amount to \$2286 in 6 yrs.?

6. Solve for P.

7. What sum of money must Robert deposit at 6% simple interest so that he will have \$1232 with which to start college 9 yrs. from now?

Accounting

Capital is the value invested in a business.

Assets include all property that belongs to the company and all money owed it.

Liabilities are the debts of the company.

Capital, assets, and liabilities are related according to the formula:

$$C = A - L$$

1. If the accountant does not use negative quantities, can you show why he puts the assets on one side of his book and the liabilities and capital on the other?

2. Find the capital of a corporation if its assets amount to \$654,000 and its liabilities amount to \$432,000.

3. A corporation, whose capital is \$128,000, has assets of \$287,000. Find the amount of its liabilities.

4. The liabilities of a corporation are \$46,000 and the capital is \$78,000. What are its assets?

The Formula in Economics

1. The average quantity of money in circulation, M, multiplied by its velocity of circulation, V, plus the average quantity of bank checking deposits, M', multiplied by their rate of turnover, V', equals the average unit price paid for goods, P, multiplied by the total volume of trade, T.

$$MV + M'V' = PT$$

(*a*) Solve this formula for P.

(*b*) Find P if $M = \$5{,}000{,}000{,}000$, $V = 20$, $M' = \$40{,}000{,}000{,}000$, $V' = 20$, and $T = \$450{,}000{,}000{,}000$.

(*c*) If the amount of money in circulation, M, were increased, other things being equal, what effect would it have on the price of goods, P?

(*d*) How would inflation, a great increase in the amount of money in circulation, affect prices?

2. The purchasing power of the farmer, P, equals the price he receives for farm products, P_f, divided by the price he pays for industrial products, P_i.

$$P = \frac{P_f}{P_i}$$

What effect will it have on the purchasing power of the farmer if:

(*a*) P_f is doubled, but P_i remains unchanged?
(*b*) P_i is doubled, but P_f remains unchanged?
(*c*) Both P_f and P_i are doubled?
(*d*) P_f is doubled, but P_i is multiplied by 3?

The Formula in Installment Payments

When you buy goods on the installment plan, you pay interest on the unpaid installments and the cost of handling them, in addition to the cash price of the goods. It is an expensive way of buying goods. The interest that is added to the cash price can be found by using the formula $i = \frac{pr(n+1)}{24}$ where n is the number of monthly installments and p is the amount unpaid. Before writing the value of p, a payment down at the beginning should be subtracted.

Illustration. A radio dealer sells a certain type of radio for \$58 cash. How much interest at 6% should he add if the terms are \$10 down and the remainder in 15 installments?

Deducting the cash payment, $p = 48$, $r = .06$, and $n = 15$

$$\begin{aligned} i &= \frac{pr(n+1)}{24} \\ &= \frac{48 \times .06 \times 16}{24} \\ &= 1.92 \end{aligned}$$

1. A real estate dealer sells houses for \$7000 cash. How much interest at 6% should he add if the buyer pays \$1000 down and the rest in 59 monthly installments?

2. Mr. Farrell borrowed \$600 which he is to pay back in 12 monthly installments with interest at 5%. How much interest will he pay?

3. The Realty Building and Loan Corporation lent \$5400 on a mortgage to be paid in 40 monthly installments with interest at 6%. What is the total amount of it?

Illustration. An automobile dealer offers a car for \$200 down and \$50 a mo. for 1 yr., or for \$735 cash. What rate of interest is he asking on his money?

You can find the rate by using the formula:

$$r = \frac{24(p + ni - c)}{n(n+1)i}$$

where r is the rate, p the first payment, n the number of installments, i the amount of each installment, and c the cash price. Then:

$$\begin{aligned} r &= \frac{24(200 + 12 \times 50 - 735)}{12 \times 13 \times 50} \\ &= .20 \end{aligned}$$

He is asking 20% interest on his money.

4. Mary Coyle bought a fur coat for \$30 down and \$10 a mo. for 8 mo. If she could have bought the coat for \$105.50 cash, what rate of interest is she paying?

5. Mrs. Eldredge bought an electric refrigerator for \$25 down and 24 monthly installments of \$5 each. The cash price was \$125. What rate of interest is she paying?

6. The Real Estate Promotion Company offers a house for \$1000 down and \$50 a mo. for 10 yrs. If the house would sell for \$2432 cash, and the rent is worth \$30 a mo., what rate of interest is the company asking?

7. The Royal Radio Company sells a certain radio for \$20 down and 12 monthly installments of \$5 each. If their cash price is \$73.50, what rate are they charging on their money? How much would I save if I borrowed the \$53.50 for a year at 6% and paid cash?

Interest by the 60-Day Method

Mr. Peabody loaned Mr. Yates \$300 for 40 days at 6% interest. To find the interest due him, Mr. Peabody uses the formula

$$i = Prt$$

$P = 300$, $r = .06$, and t is the time in years. He could express the time as $\frac{40}{365}$ of a year, but this would be inconvenient, so in agreement with other bankers he uses 360 days as 1 yr.

For 360 days, $i = .06\,P$.
For 60 days ($\frac{1}{6}$ of 360), $i = .01\,P$.

Now to multiply by .01 we move the decimal point two places to the left. This is most conveniently done by drawing a line down to represent the decimal point in all the numbers in a column.

To find the interest for 40 days, divide 40 into numbers that are factors of 60, as 30 and 10. As 30 days is half of 60 days, divide 3.00 by 2 and as 10 days is $\frac{1}{6}$ of 60 days, divide 3.00 by 6. Then the sum of the interest for 30 days and for 10 days is the interest for 40 days.

3	00	60 days
1	50	30 days
	50	10 days
2	00	40 days

Illustration. Find the interest on $480 for 77 days at 6%.

Factors of 60 that make 77 are 60, 15, and 2. For 15 days divide by 4. For 2 days divide by 30. To do this, divide by 3, and move the decimal point one place to the right.

4	80	60 days
1	20	15 days
	16	2 days
6	16	77 days

Class Exercises

Find the interest at 6% on:

	Principal	*No. of Days*		*Principal*	*No. of Days*
1.	$200	30	**11.**	$ 740.20	20
2.	500	45	**12.**	86.50	24
3.	450	50	**13.**	4362.40	80
4.	600	22	**14.**	1145	12
5.	720	36	**15.**	28.67	90
6.	870	42	**16.**	2463.80	77
7.	64	70	**17.**	624.60	130
8.	954	52	**18.**	583.10	14
9.	840	47	**19.**	82.13	17
10.	420	28	**20.**	1691.43	243

Optional Exercises

When the rate is not 6%, find the interest first at 6%, and then correct it for the proper rate.

Illustration. Find the interest on $643.55 for 28 days at 4½%.

6	4355	60 days	at 6%
3	2178	30 days	
	2145	2 days	
3	0033	28 days	at 6%
	7508		at 1½%
2	2525		at 4½%

$2.25 is the answer.

To find the interest for 28 days, find it for 30 and 2, and subtract.

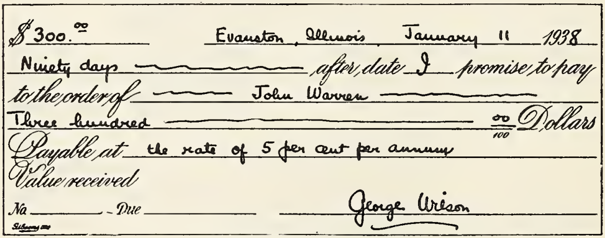
Divide by 4, since 1½% is ¼ of 6%. Then subtract.

Keep extra decimal places to the end, and then adjust to the nearest cent.

	Principal	*Time*	*Rate*		*Principal*	*Time*	*Rate*
21.	$ 420	40 d.	4½%	**26.**	$ 166.40	2 mo.	7%
22.	335	70 d.	4½%	**27.**	486.12	3 mo.	8%
23.	251	38 d.	5%	**28.**	2100.40	5 mo.	5%
24.	6540	50 d.	3%	**29.**	58.25	7 mo.	4%
25.	63	46 d.	4%	**30.**	554.36	1 mo.	2%

BANK DISCOUNT

Business men, whose credit is good, often give a promissory note in payment of a bill. This is a promise to pay a sum of money in a certain number of days, usually 30, 60, or 90, or in a certain number of months. If the man who receives this note needs the money before the note will be paid, he takes it to a bank. The banker deducts interest in advance for the time he must wait for the money and gives the man the remainder.



$300.⁰⁰ Evanston, Illinois, January 11 1938

Ninety days after date I promise to pay

to the order of John Warren

Three hundred 00/100 Dollars

Payable at the rate of 5 per cent per annum

Value received

No. — Due — George Wilson

A PROMISSORY NOTE

The date on which he takes the note to the bank is the *date of discount;* the date when the note must be paid by the man who wrote it is the *date of maturity;* and the time between these two dates is the *term of discount.* The date on which the note was written is the *date of the note,* and the amount of money it promises to pay is the *face of the note.*

How to find the date of maturity. If the time is given in months, you simply count the required number of months ahead and take the same day of that month. For example, a note dated April 18 promising to pay in 3 mo. would mature July 18. If the time is stated in days, you must count the exact number of days from the date of the note. For example, a 60-day note dated June 3 would be due August 2 (27 days left in June, 31 in July, and 2 needed from August to make 60).

How to find the term of discount. Count the exact number of days from the date of discount to the date of maturity. For example, if a note due August 2 was discounted June 11, the term would be 52 days (19 left in June, 31 in July, and 2 in August).

Bankers often use a table for finding the time between dates.

BANKERS' TIME TABLE

FROM ANY DAY OF	TO THE SAME DAY OF THE NEXT											
	Jan.	*Feb.*	*Mar.*	*Apr.*	*May*	*Jun.*	*Jul.*	*Aug.*	*Sept.*	*Oct.*	*Nov.*	*Dec.*
Jan.	365	31	59	90	120	151	181	212	243	273	304	334
Feb.	334	365	28	59	89	120	150	181	212	242	273	303
Mar.	306	337	365	31	61	92	122	153	184	214	245	275
Apr.	275	306	334	365	30	61	91	122	153	183	214	244
May	245	276	304	335	365	31	61	92	123	153	184	214
Jun.	214	245	273	304	334	365	30	61	92	122	153	183
Jul.	184	215	243	274	304	335	365	31	62	92	123	153
Aug.	153	184	212	243	273	304	334	365	31	61	92	122
Sept.	122	153	181	212	242	273	303	334	365	30	61	91
Oct.	92	123	151	182	212	243	273	304	335	365	31	61
Nov.	61	92	120	151	181	212	242	273	304	334	365	30
Dec.	31	62	90	121	151	182	212	243	274	304	335	365

This table gives the time from any day of a month on the left to the same day of the month at the top of a column. To find the time from August 5 to December 5, find August on the left side of the table. Then look across on the same line to the column headed December. The time is 122 days.

To find the time from May 8 to September 20, find May on the left and look across to the column headed September. The 123 days given there is the time from May 8 to September 8. As there are 12 days more from September 8 to September 20, add 12 to the 123. The time then is 135 days.

To find the time from October 29 to March 18, we find 151 days from October 29 to March 29. As March 18 is 11 days earlier, subtract 11. The time is 140 days.

Exercises

Find the date of maturity and the term of discount for these notes:

	Date of Note	*Time*	*Date of Discount*
1.	Apr. 27	4 mo.	May 10
2.	Jan. 25	2 mo.	Feb. 1
3.	Sept. 5	3 mo.	Sept. 19
4.	Jun. 24	3 mo.	Aug. 27
5.	Jan. 2	4 mo.	Mar. 30
6.	Jun. 3	60 d.	Jul. 10
7.	Dec. 14	90 d.	Jan. 4
8.	Oct. 29	30 d.	Nov. 8
9.	May 20	60 d.	May 28
10.	Jul. 2	90 d.	Jul. 19

How to find the proceeds of a note. The amount of money left after the discount is deducted from the face of the note is called the ***proceeds***. To find the proceeds, we must first find the discount or interest on the face of the note for the term of discount. Then we subtract this discount from the face of the note.

$$p = f - fdt$$

where p is the proceeds, f the face of the note, d the rate of discount, and t the time in years.

Illustration. Find the proceeds of a 60-day note for $240 dated April 23 and discounted May 5 at 6%.

Date of maturity (60 days after April 23) is June 22.

Term of discount (from May 5 to June 22) is 48 days.

Discount on $240 for
48 days at 6%.

2	40	60 d.
	48	12 d.
1	92	48 d.

Proceeds = face − discount

Face	240.00
Discount	1.92
Proceeds	238.08

Exercises

Find the proceeds of these notes:

	Face	*Date*	*Discounted*	*Rate*	*Time*
1.	$ 400	Jul. 2	Jul. 20	6%	60 d.
2.	350	Aug. 27	Sept. 5	6%	30 d.
3.	420	Jan. 2	Feb. 1	6%	90 d.
4.	140	Mar. 4	Mar. 4	6%	30 d.
5.	3460	May 12	Jun. 24	6%	3 mo.
6.	240.65	Oct. 29	Nov. 20	4½%	2 mo.
7.	65.49	Jul. 19	Aug. 4	5%	4 mo.
8.	343.44	Feb. 20	Mar. 1	4%	1 mo.
9.	1243.66	Apr. 27	May 6	3%	2 mo. 15 d.
10.	6543.21	Mar. 30	May 1	5½%	5 mo. 20 d.

COMPOUND INTEREST

In a savings bank interest is computed to certain dates, usually January 1 and July 1. If you leave the interest in the bank, it is added to the principal, and the interest for the next period is computed on this larger principal, and so on. When interest is added once a year, it is said to be compounded annually. When it is added every half year, it is compounded semi-annually; and when it is added every three months, it is compounded quarterly. When interest is compounded semi-annually, the rate for one period is half the annual rate.

Illustration. Find the amount of $420 for 3 yrs. at 6% compounded annually.

Principal at beginning	$420.
Interest first yr.	25.20
Amount at end of 1 yr.	445.20
Interest second yr.	26.712
Amount at end of 2 yrs.	471.912
Interest third yr.	28.3147
Amount at end of 3 yrs.	500.2267
Answer to the nearest cent	$500.23

Exercises

Find the amount of:

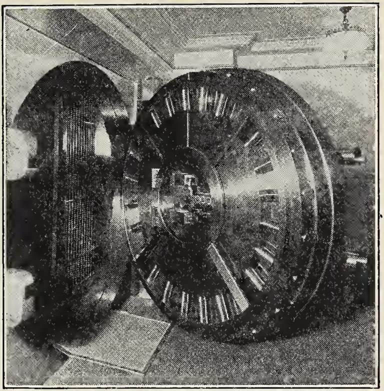
1. $300 for 2 yrs. at 6% compounded annually
2. $540 for 3 yrs. at 6% compounded annually
3. $200 for 2 yrs. at 6% compounded semi-annually
4. $784 for 2½ yrs. at 6% compounded semi-annually
5. $100 for 2 yrs. at 4% compounded quarterly

How to make the compound-interest formula. You can see that the work of finding the compound interest for a long period of time would be very great. Here is a short cut that reduces the labor to a single multiplication. Later you will learn another short method that can be used for any rate and any time.

You have already learned that you can make a general formula by using letters in place of all numbers and solving the problem. Let us then find the amount of P dollars at rate r for n yrs. compounded annually.

Principal at beginning	P
Interest first yr.	Pr
Amount at end of 1 yr.	$P + Pr$
Factoring	$P(1 + r)$

Notice that $P(1 + r)$ shows that we would have gotten the same result if, instead of finding the interest and adding



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THE DOOR OF A BANK VAULT

it to the principal, we had multiplied the principal by $1 + r$. In that case since $P(1 + r)$ becomes the new principal on which the next year's interest is computed, we ought to be able to get the amount at the end of the second year by multiplying by $1 + r$ again, and so on. That would give us:

Principal at beginning	P
Amount at end of 1 yr.	$P(1 + r)$
Amount at end of 2 yrs.	$P(1 + r)^2$
Amount at end of 3 yrs.	$P(1 + r)^3$

What would be the amount at the end of 4 yrs.? Of 5 yrs.? Of 10 yrs.? Of n yrs.?

The formula for the amount of P dollars at rate r for n yrs. compounded annually is:

$$A = P(1 + r)^n$$

How to use a compound-interest table. A compound-interest table gives the values of $(1 + r)^n$, so it is only necessary to multiply by P to find the amount of P dollars.

To find $(1 + r)^n$, look down the column on the left, headed n, until you come to the value you need. Then look across that line to the column under the proper value of r.

Illustration. Find the amount of \$750 for 8 yrs. at 5% compounded annually. Here $P = 750$, $n = 8$, and $r = 5\%$. Look down the column headed n until you come to 8. Then follow that line across to the column headed 5%. You will find 1.477455. This is $(1 + r)^n$. Multiplying by $P = 750$, we have $A = 1108.09125$ or \$1108.09.

COMPOUND INTEREST

Values of $(1 + r)^n$

n	*1%*	*2%*	*3%*	*4%*	*5%*	*6%*
1.	1.010000	1.020000	1.030000	1.040000	1.050000	1.060000
2.	1.020100	1.040400	1.060900	1.081600	1.102500	1.123600
3.	1.030301	1.061208	1.092727	1.124864	1.157625	1.191016
4.	1.040604	1.082432	1.125509	1.169859	1.215506	1.262477
5.	1.051010	1.104081	1.159274	1.216653	1.276282	1.338226
6.	1.061520	1.126162	1.194052	1.265319	1.340096	1.418519
7.	1.072135	1.148686	1.229874	1.315932	1.407100	1.503630
8.	1.082857	1.171659	1.266770	1.368569	1.477455	1.593848
9.	1.093685	1.195093	1.304773	1.423312	1.551328	1.689479
10.	1.104622	1.218994	1.343916	1.480244	1.628895	1.790848

Class Exercises

If interest is compounded annually, find the amount of:

	P	r	n		P	r	n
1.	\$100	6%	5	**8.**	\$8421.50	4%	4
2.	452	6%	9	**9.**	3175.68	5%	10
3.	300	4%	8	**10.**	7436.54	3%	8
4.	560	3%	6	**11.**	2107.09	6%	6
5.	287	5%	7	**12.**	6342.50	2%	10
6.	194	5%	10	**13.**	5891.65	6%	7
7.	891	2%	8	**14.**	3230.17	5%	9

Optional Exercises

When interest is compounded semi-annually, let n *be the number of interest periods (twice the number of years), and let* r *be the rate for one period* ($\frac{1}{2}$ *the annual rate).*

If interest is compounded semi-annually, find the amount of:

	Principal	*Rate*	*Time*		*Principal*	*Rate*	*Time*
15.	$100	6%	3 yrs.	**20.**	$673.42	6%	3 yrs. 6 mo.
16.	300	4%	4 yrs.	**21.**	211.54	4%	2 yrs. 6 mo.
17.	685	2%	5 yrs.	**22.**	583.90	6%	4 yrs. 6 mo.
18.	492	6%	3 yrs.	**23.**	87.64	4%	2 yrs. 6 mo.
19.	147	4%	2 yrs.	**24.**	786.59	2%	4 yrs. 6 mo.

Honor Work

When the time is not an exact number of periods, find compound interest for the greatest number of full periods, and then find simple interest on this amount for the part period.

25.	$100	4%	3 yrs. 8 mo.	Semi-annually
26.	584	5%	8 yrs. 9 mo.	Annually
27.	643	4%	2 yrs. 7 mo.	Quarterly
28.	967	6%	4 yrs. 10 mo.	Semi-annually

29. In how many years will $100 at 5% compounded annually amount to $134?

Review Exercises

1. Find the interest on $842 at 6% for 42 days.

2. Find the amount of $900 at 5% compounded annually for 7 yrs.

3. A note dated May 12 and payable in 90 days is discounted June 18.

(*a*) Find the date of maturity and the term of discount.

(*b*) If the face of the note is $320 and the rate of discount 6%, find the proceeds.

4. The fundamental equation in accounting is $A - L = C$. What change takes place in the:

(*a*) Capital, if the liabilities increase, but the assets remain unchanged?

(*b*) Capital, if the assets increase, but the liabilities remain unchanged?

(*c*) Liabilities, if the capital increases, but the assets remain unchanged?

(*d*) Assets, if the capital increases, but the liabilities remain unchanged?

5. Use the percentage formula $p = br$.
(*a*) 18 is what per cent of 90?
(*b*) 48 is 12% of what number?
(*c*) Find 15% of 80.

6. How much money must I place at 6% interest compounded annually so that I shall have \$1000 in 8 yrs.?

7. Solve for x and y:

$$\begin{aligned} ax + by &= b \\ dx + y &= 1 \end{aligned}$$

8. (*a*) Solve for x: $mx + k = x - 5$

(*b*) By using your answer as a formula, solve:

$$5x + 3 = x - 5$$

9. (*a*) Solve for x: $ax + b = nx + r$

(*b*) Using your answer as a formula, solve:

$$7x + 5 = 5x + 11$$

10. (*a*) Two automobiles d mi. apart are traveling toward each other at rates of r_1 and r_2 mi. an hr. In how many hours will they meet?

(*b*) Using your answer as a formula, find the time if $d = 260$, $r_1 = 25$, and $r_2 = 40$.

(*c*) What is the effect on the time: If d only is increased? If r_1 only is increased? If d, r_1, and r_2 are all doubled? If d is doubled, but r_1 and r_2 are each multiplied by 3?

11. (*a*) Henry Jones invested P dollars, part at rate b and part at rate c. His income is d dollars a yr. How much has he at each rate?

(*b*) Using your answer to part *a* as a formula, find the amounts at each rate if Mr. Jones invested \$6000, part at 5% and part at 6%, and his income was \$310.

12. (*a*) How much water must Sam add to z oz. of a p% solution of alcohol to reduce it to an n% solution?

(*b*) Using the answer to part *a*, find how much water he must add to 16 oz. of a 40% solution to reduce it to a 15% solution.

13. How much must I invest at 5% to yield an income of \$1200 annually? \$2400 annually? To double the income must you double the investment?

14. The population of S increased 15% in the last 10 yrs. and is now 43,700. What was the population of the town 10 yrs. ago?

15. Robert Brown sold his car for \$580 at a loss of $33\frac{1}{3}\%$ of the cost. How much did he pay for it?

16. At what price must Cohen Brothers sell a suit bought for \$16.80 to make 40% of the selling price?

17. In the compound-interest table, when n increases, does $(1 + r)^n$ increase or decrease?

18. If you subtract 1, which is the principal, from each value of $(1 + r)^n$, the remainder is the interest. When $n = 10$, is the interest just 10 times what it was when $n = 1$?

19. As r increases, does $(1 + r)^n$ increase or decrease? When $r = 6$, is the interest just 6 times the interest when $r = 1$ (*a*) if $n = 1$? (*b*) If $n = 10$?

Solve these formulas for the letter following the formula:

20. $fd = w$ $\quad f$
21. $C = 2\pi r$ $\quad r$
22. $p = 2l + 2w$ $\quad w$
23. $C = \frac{E}{R}$ $\quad E$
24. $V = \frac{bh}{3}$ $\quad h$
25. $v^2 = 2gs$ $\quad s$
26. $S = 2\pi rh$ $\quad r$
27. $v = 20 + 32t$ $\quad t$
28. $D = dq + r$ $\quad q$
29. $\frac{w_1}{w_2} = \frac{d_2}{d_1}$ $\quad w_2$
30. $f = \frac{Pl^3}{3EI}$ $\quad P$
31. $S_1 = \frac{Mc_1}{I_1 + nI_2}$ $\quad M$

Solve for x:

32. $kx = hk$
33. $\pi x = 11\pi$
34. $acx = abc$
35. $\frac{x}{a} = 3b$
36. $\frac{x}{a + b} = 1$
37. $\frac{c}{x} = 5$
38. $\frac{r}{x} = s$
39. $ax - a = 0$
40. $nx + an = 0$
41. $\frac{ax}{b} = a$
42. $\frac{rx}{s} = t$
43. $nx = n^2$
44. $a = bx$
45. $c = \frac{h}{x}$
46. $ax + bx = a + b$
47. $4x + 8a = 12b$
48. $3x + m = x + 7m$
49. $2px - a = px + 3a$
50. $2(x + 2b) = 10b$

51. $3(x - a) = 2(x + a)$

52. $\frac{x}{a} = \frac{b}{c}$

53. $\frac{x - n}{b} = m - \frac{n}{b}$

54. $\frac{a}{x} + a = \frac{2a}{x}$

55. $a(a - x) = b(b - x)$

56. $\frac{x}{p} + \frac{x}{q} = 1$

57. $\frac{2x}{r} - \frac{x}{s} = v$

58. $\frac{b}{x - a} = \frac{3b}{2a}$

59. $\frac{x}{a} + \frac{x}{b} - \frac{x}{c} = d$

Test in Literal Equations: One Unknown

Solve for x:

1. $3x = b$

2. $ax = b$

3. $5kx = m$

4. $\frac{x}{5} = a$

5. $\frac{x}{d} = q$

6. $x + a = 7$

7. $2x - b = r$

8. $5x - 3n = k$

9. $ax + 5m = 3c$

10. $\frac{x}{d} + f = g$

11. $ax + b = 3cx - d$

12. $\frac{x}{a} + \frac{x}{2} = 10$

13. $\frac{x}{m} - \frac{x}{p} = s$

14. $\frac{x}{5} - \frac{x - 1}{c} = a$

15. $\frac{nx + 2}{n} - \frac{kx - 1}{2k} = h$

Test in Literal Equations: Two Unknowns

Solve for x and y:

1. $x + y = a$
$x - y = 5$

2. $ax + 2y = b$
$x + y = c$

3. $3x + 4y = k$
$4x - 3y = m$

4. $mx + py = q$
$5x + ry = 7$

5. $ax + by = c$
$x = dy$

6. $(a + 1)x + 2y = c$
$(a - 1)x - 3y = d$

Test in Solving Formulas

Solve for the letter following the formula:

1. $L = 4as$ $\quad s$

2. $p = 2a + 2b$ $\quad b$

3. $F = \frac{mv^2}{r}$ $\quad r$

4. $R = \frac{3pm}{2wh^2}$ $\quad p$

5. $P = a + dgh$ $\quad d$

6. $\frac{PV}{T} = \frac{pv}{t}$ $\quad v$

7. $C = \frac{e}{R + r}$ $\quad r$

8. $s = \frac{rl - a}{r - 1}$ $\quad l$

9. $c = \frac{h_2 - h_1}{h_1(t_2 - t_1)}$ $\quad h_2$

10. $a = \frac{(m_2 - m_1)g}{m_2 + m_1}$ $\quad m_2$

Chapter 11

SQUARE ROOT AND RADICALS

JOHN and Robert found the height of a cliff by dropping a stone and noting the time it took to reach the bottom. They used the formula $d = 16\,t^2$, substituting the number of seconds for t and getting the height d in feet. Robert then asked, "How long would it take a stone to fall a mile?" "That is easy," said John. "We shall substitute 5280 ft. for d and solve for t." He got this result:

$$\begin{aligned} 16\,t^2 &= d \\ 16\,t^2 &= 5280 \\ t^2 &= 330 \\ t &= \sqrt{330} \end{aligned}$$

John wrote $t = \sqrt{330}$, meaning the square root of 330, for finding the square root is the opposite of squaring, and he knew that to get rid of a square he should do its opposite. Then he found the square root of 330. Could you have done it? Let us see how it is done.

Investigating square root. How many digits are there in the square of a one-digit number? Of a two-digit number? Of a three-digit number?

$$\begin{aligned} 1^2 &= 1 \\ 9^2 &= 81 \\ 10^2 &= 1'00 \\ 99^2 &= 98'01 \\ 100^2 &= 1'00'00 \\ 999^2 &= 99'80'01 \end{aligned}$$

If we group the digits in the square by two's beginning at the decimal point, how does the number of groups compare with the number of digits in the square root?

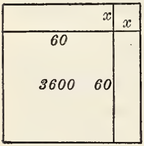
Notice that if we separate the square into groups of two digits each, beginning at the decimal point, each group will represent a digit in the square root.

How to find the square root of a number.

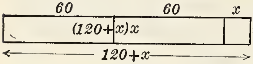
Illustration. Find the square root of 3969.

First begin at the decimal point, and separate the number into groups of two digits each. The square root of 39′69 has two digits.

Since 39 is between 36 and 49 (the squares of 6 and 7) the square root of 39′69 is between 60 and 70. Call it $60 + x$.



Look at the square. It is made up of a large square, two rectangles, and a small square. If we subtract the area of the large square, $60^2 = 3600$ from 39′69, we have 369 left for the two rectangles and the small square.



Laying the three pieces out to form a single rectangle, we note that its area is 369 and that its length is a little more than **twice** the side of the square or 120. So we can approximate the width by dividing the area remaining by the length which is twice the answer already found. $369 \div 120 = 3 +$. Then we find the exact length of the rectangle by adding this 3 to 120. Since $123 \times 3 = 369$, x is exactly 3, and the side of the square, $60 + x$, is 63.

We can arrange the work as follows:
Subtract $60^2 = 3600$ from 3969:
The trial divisor is twice 60:

```
           6
         √39'69
          36 00
    120  | 3 69
```

Since 120 goes into 369 3 times, add 3 to the answer making the answer 63:
Add 3 to the trial divisor making it 123:
Multiply the divisor by the new answer, 3, and subtract:

```
          6 3
         √39'69
          36 00
    123  | 3 69
           3 69
```

Since there is no remainder, the square root is exactly 63.

As in long division, unnecessary 0's, such as those after the 36 and the 0 of the 120 may be omitted.

Thought Question

Is the square root of 1690 exactly 10 times the square root of 169? Explain.

Exercises

Find the square root of:

1. 1936
2. 9409
3. 3136
4. 1024
5. 6724
6. 29.16
7. 5.76
8. .8649
9. 12.96
10. .000841
11. 701.7201
12. 26.2144
13. 466.56
14. .351649
15. 1176.49

When a number is not a perfect square, its square root will not come out even, but you can find it to as many decimal places as needed by adding enough 0's after the decimal point to make a group of two figures for each decimal place in the number.

Find these square roots to three decimal places:

16. 2
17. 3
18. 5
19. 11
20. 39
21. 53
22. 38.47
23. 4.912
24. .617
25. 6.4
26. 5.8
27. .58

Thought Questions

Can the same number have more than one square root? What number is the square root of 49? Is there any number other than 7 that, when multiplied by itself, gives 49? How many square roots then has 49?

However, to avoid confusion, we shall agree that when we write the form $\sqrt{a}$, we shall mean the positive square root only: that is, $\sqrt{25}$ is $+5$.

APPROXIMATE MEASUREMENT

How accurately can you measure? Measure the length and width of the picture showing the transit on page 310, and read your result to just as small a fraction of an inch as you are able. Now find the area of the picture. Do not let anyone know your answer until all members of the class have finished. Now, do all of you get the same result? If not, which of you is right, or are all wrong? On how many figures of your answer do most of you agree? What then is the use of all the other figures if they are probably wrong? Would not the figures of which you are sure, followed by 0's, be just as accurate?

You see from this that measurement is not exact. We can think of a line segment as exactly 8 in. long or as exactly equal to another segment, but because we are unable to make instruments that are absolutely accurate, and to read them beyond a certain degree of exactness, the real length of a segment can never be known. All physical measurements are only approximately true. With an ordinary ruler we cannot measure the length of this page more accurately than to the hundredth of an inch. We read the length as 7.25 in. if, in our judgment, it is more than 7.245 and less than 7.255, that is, if it is nearer to 7.25 than to 7.24 or 7.26. But we cannot read accurately enough to tell if it is 7.249 or 7.252. Therefore, we say that the length is 7.25 in. to the nearest hundredth of an inch.

If we used a more accurate instrument than the ruler, we might determine that the length was 7.252 and not 7.251 or 7.253, but we would still be unable to tell whether it differed from 7.252 by 1 or 2 ten-thousandths of an inch. All that our more accurate instrument can do is to move the uncertainty to a different decimal place.

Significant figures. Figures obtained by actual measurement which indicate the degree of accuracy of the measurement are called ***significant figures.*** Sometimes 0 is significant. When we measured with the more accurate instrument, if we had found that the result was nearer to 7.25 than to 7.251 or to 7.249, we would have written the result 7.250 in. This 0 does not change the result, but it shows that we have measured the length more accurately than to the hundredth of an inch. Consequently it is significant. In the first measurement, the 7, 2, and 5 are all significant figures, for the result is accurate to the hundredth of an inch, but if 0's were written after the 5, they would not be significant.

More often, however, 0's are not significant figures, but are used only to determine the decimal point. When a man states, for example, that he expects to save $1000 next year, he does not mean that he will not save $1,143.57, but only that he does not expect to differ from $1000 by many hundred dollars. In this case, the 0's merely determine the decimal point; they show that he does not expect his savings to be around $10,000 or $100. So the number 1000 has only one significant figure. Similarly, the distance from the earth to the moon, 238,000 mi., has only three significant figures, for it may differ from the exact distance by 300 or 400 mi. A number that has 0's to determine the decimal point is called a *round number.*

All figures except 0's are usually significant. We are accustomed to say that a distance is about 200 mi. when we know it is 198 mi., but we do not say it is about 198 mi. when we know it is 200 mi.

Computation with approximate numbers. If we find by measurement that the side of a square is 5.4 in., correct to the nearest tenth of an inch, then the diagonal will be $5.4\sqrt{2}$. Now $\sqrt{2} = 1.41421$ correct to five decimal places, which gives us 7.636734 as the length of the diagonal. However, 5.4 in. to the nearest tenth of an inch tells us only that the side of the square is between 5.35 and 5.45 in., and may differ from the exact length as much as .04 in. The length of the diagonal then, computed from this number, may differ from the true length by more than .04 in. Consequently it would be not only useless but misleading to give the result 7.636734 when even the 3 in the hundred's place may be incorrect. Since 5.4 is accurate to only two significant figures, the length of the diagonal cannot be depended on to more than two significant figures. As a rule, in working with approximate measurements, we carry the result to only as many significant figures as the least accurate of the measurements used in obtaining it.

How to round off a number. When we express a number to less significant figures than are given, we call it rounding off the number. When the figure to be omitted is less than 5, we simply drop it, but leave the other figures as they were (or if on the left of the decimal point, we replace it by a 0). When, however, the figure to be omitted is 5 or more, we increase the last figure retained by 1. For example, the value of π to 7 significant figures is 3.141593. To six significant figures it is 3.14159, for since the 3 is less than 5, we drop it. To five significant figures the value is 3.1416, for since the 9 is more than 5, we increase the 5 to a 6.

Exercises

Express these numbers to three significant figures:

1. 25.83
2. 3.164
3. 59.66
4. 487.8
5. 631.3
6. .4885
7. 5932
8. 80471
9. 24333
10. .7699
11. 49999
12. 62.99

In these exercises draw a line under 0's that are significant:

13. 1020
14. .001
15. 0.034
16. 5.003
17. 1000
18. 24.0
19. .050
20. 30.0

21. Is 0 significant in the date 1940?

22. Is 0 significant in $5 \times 4 = 20$?

23. John measured the length of his room and found it to be 12.84 ft. to the hundredth of a foot. Robert found the width to the nearest foot to be 11 ft. What was the area of their room?

24. On a map, the distance from *A* to *B* is 2.4 in. and from *B* to *C* 4.8 in. If the distance from *A* to *B* is 267.6 mi., what is the distance from *B* to *C*?

Thought Questions

1. The value of π is 3.14159265. How many significant figures of it would be reasonable in finding the circumference of a circle whose radius is:

(*a*) 5.41? (*b*) 103.7? (*c*) 204.0? (*d*) 124.537?

2. Would you round off the answer in finding the cost of 3847 books at $.93 each?

SQUARE ROOT AND THE FORMULA

Class Exercises

1. (*a*) Solve $A = s^2$ for s.

(*b*) Find s to the nearest hundredth when $A = 43.4$.

2. The pressure on a dam is $p = 32\,wd^2$ where w is the width of the river and d its depth.

(*a*) Solve for d.

(*b*) Find d to three significant figures if $w = 350$ and $p = 105{,}000$.

3. The velocity of a falling body is given by $v^2 = 2\,gh$, where $g = 32$ and h is the height fallen.

(*a*) Solve for v.

(*b*) Find v to three significant figures when $h = 1200$ and $g = 32$.

4. The number of watts of electricity used is $w = \frac{v^2}{r}$, where v is the voltage and r the resistance.

(*a*) Solve for v.

(*b*) Find v to two significant figures when $w = 61$ and $r = 370$.

(*c*) If v is constant, how does w change as r increases?

5. Another electricity formula is $w = c^2r$.

(*a*) Solve for c.

(*b*) Find c to tenths when $w = 330$ and $r = 800$.

(*c*) If c is constant, how does w change as r increases?

6. (*a*) Solve $d = \frac{1}{2}\,at^2$ for t.

(*b*) Find t to the nearest unit when $a = 32.4$ and $d = 10{,}560$.

7. (*a*) Solve $A = 6\,e^2$ for e.

(*b*) Find e to as many significant figures as A has, if $A = 7.340$.

8. The volume of a square prism is $V = s^2h$.

(*a*) Solve for s.

(*b*) Find s to tenths if $V = 55$ and $h = 5$.

Optional Exercises

Find the answers to two significant figures:

9. In $A = \pi r^2$, find r when $A = 616$ and $\pi = \frac{22}{7}$.

10. In $V = \pi r^2h$, find r when $\pi = 3.14$, $h = 11$, and $V = 628$.

11. In $c^2 = a^2 + b^2$, find a when $c = 38$ and $b = 23$.

12. In $x^2 + y^2 = 22$, find y when (*a*) $x = 0$, (*b*) $x = 1$, (*c*) $x = 2$, (*d*) $x = 3$, (*e*) $x = 4$. (*f*) Plot these points on a graph. How does the point (x, y) seem to move as x increases?

13. In $E = \frac{1}{2} mv^2$, find v when $E = 34$ and $m = 16$.

14. The slant height s of a cone is $s^2 = h^2 + r^2$.

(*a*) Solve for h.

(*b*) Find h when $r = 9.0$ and $s = 14$.

Honor Work

15. Any two bodies in the universe attract each other with a force $f = \frac{kw_1w_2}{d^2}$, where $k = .00\,000\,000\,000\,014$, w_1 and w_2 are the weights of the bodies, and d is the distance between their centers.

SATURN AND ITS RINGS

These rings are probably made of small particles held in their course around the planet by the force of gravitation.

(*a*) Solve for d.

(*b*) Find d when $w_1 = 2{,}000{,}000$, $w_2 = 30{,}000{,}000$, and $f = .004$

(*c*) Look at the original formula. As d increases, does f increase or decrease?

(*d*) Is there any body so far from the earth that the force exerted on it by the earth is exactly 0?

16. One answer of a quadratic equation, an equation containing x^2, can be found by using the formula:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

Substitute the following values in the formula, and find x, to the nearest tenth when it does not come out an even number:

(*a*) $a = 2, b = 5, c = 2$ (*c*) $a = 1, b = 11, c = -5$

(*b*) $a = 3, b = 8, c = 2$ (*d*) $a = 3, b = -2, c = -6$

If in this formula, a is the coefficient of x^2, b the coefficient of x, and c the number, solve:

(*e*) $x^2 + 3x + 2 = 0$ (*g*) $2x^2 - 6x - 5 = 0$

(*f*) $x^2 + 2x - 3 = 0$ (*h*) $3x^2 - 8x + 4 = 0$

17. When money is placed at compound interest, the amount in 2 yrs. is $A = P(1 + r)^2$.

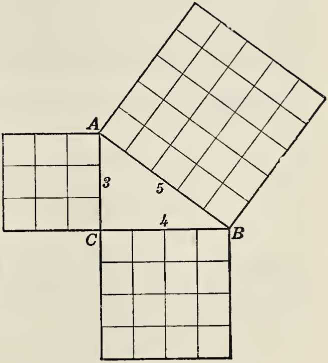
(*a*) Solve for r. (Do not multiply out the parenthesis.)

(*b*) Find r if $P = 350$ and $A = 378.56$.

18. A formula used to find π is $x = \sqrt{2r^2 - r\sqrt{4r^2 - s}}$. Find x to 2 significant figures if $s = 1$ and $r = 1$.

THE PYTHAGOREAN THEOREM

Triangle *ABC* is a right triangle. It has a right angle at *C*. In this particular triangle, the sides are 3 in., 4 in., and 5 in. What is the area of the square on 3? On 4? On 5? If you add the squares on sides 3 and 4, how does the sum compare with the square on 5?



A triangle is called a *right triangle* when one of its angles is a right angle. The side opposite the right angle, the longest side of the triangle, is called the ***hypotenuse,*** and the other two sides are the ***legs.*** In triangle ABC, AB or c is the hypotenuse. AC and BC, or b and a, are the legs.

PYTHAGORAS

Pythagoras (580–501 B.C.) was a famous Greek mathematician. He formed a secret society in southern Italy which worked out many of the theorems of geometry long before Euclid wrote his epochal book. Although special cases of the theorem that bears his name were used in practical measurement long before his time, he was the first to prove that it was true for all cases.

The relationship that you found true for the 3, 4, 5 triangle, is true for all right triangles. It was known to the ancient Egyptians, but was first proved by a Greek mathematician Pythagoras about the year −550.

In a right triangle, the square of the hypotenuse equals the sum of the squares of the legs.

$$c^2 = a^2 + b^2$$

Illustration 1: Find the hypotenuse of a right triangle whose legs are 8 and 15.

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= 8^2 + 15^2 \\ &= 64 + 225 \\ &= 289 \end{aligned} \qquad \begin{aligned} c &= \sqrt{289} \\ &= 17 \end{aligned}$$

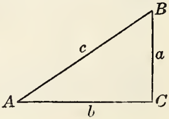


Illustration 2: Find the leg of a right triangle whose hypotenuse and other leg are 82 and 80.

$$\begin{aligned} a^2 &= c^2 - b^2 \\ &= 82^2 - 80^2 \end{aligned} \qquad \begin{aligned} a &= \sqrt{324} \\ &= 18 \end{aligned}$$

Exercises

Find the hypotenuse of a right triangle whose legs are:

1. 6 and 8
2. 5 and 12
3. 21 and 28
4. 7 and 24
5. 5 and 7
6. 10 and 10
7. 38 and 43
8. 63 and 57
9. 184 and 362
10. Solve the formula $c^2 = a^2 + b^2$ for a, for b.

Find the other leg of a right triangle if the hypotenuse and one leg are:

11. 15 and 9
12. 26 and 10
13. 41 and 9
14. 184 and 138
15. 111 and 94
16. 342 and 341
17. 3.96 and 3.12
18. .083 and .075
19. 76.0 and 63.4

The table of square roots. Mr. Brown is a machinist, and uses squares and square roots very often in his business. It would not pay him to stop to work out a square root every time he needed one, so he uses a table like that on page 254. To find the square root of 43, he looks down the column headed *No.* until he comes to 43. Then he looks across to the column headed *Square Roots.* Here he finds that the square root of 43 is 6.557 to four significant figures.

Exercises

By using the table, find the square root of:

1. 5
2. 8
3. 57
4. 72
5. 50
6. 83
7. 31
8. 95

9. Can Mr. Brown find the square root of .24 from this table? If the first group of two figures is just right of the decimal point, where would you place the decimal point in its square root?

10. To find the square root of .8, would you look opposite 8 or 80? Why? What is the square root of .8?

11. Using the table, find the square root of 4100.

12. How does the square root of 8 compare with that of 2? Can you give a reason for your answer.

Table of Squares and Square Roots

No.	*Squares*	*Square Roots*	*No.*	*Squares*	*Square Roots*	*No.*	*Squares*	*Square Roots*
1	1	1.000	**34**	1,156	5.831	**67**	4,489	8.185
2	4	1.414	**35**	1,225	5.916	**68**	4,624	8.246
3	9	1.732	**36**	1,296	6.000	**69**	4,761	8.307
4	16	2.000	**37**	1,369	6.083	**70**	4,900	8.367
5	25	2.236	**38**	1,444	6.164	**71**	5,041	8.426
6	36	2.449	**39**	1,521	6.245	**72**	5,184	8.485
7	49	2.646	**40**	1,600	6.325	**73**	5,329	8.544
8	64	2.828	**41**	1,681	6.403	**74**	5,476	8.602
9	81	3.000	**42**	1,764	6.481	**75**	5,625	8.660
10	100	3.162	**43**	1,849	6.557	**76**	5,776	8.718
11	121	3.317	**44**	1,936	6.633	**77**	5,929	8.775
12	144	3.464	**45**	2,025	6.708	**78**	6,084	8.832
13	169	3.606	**46**	2,116	6.782	**79**	6,241	8.888
14	196	3.742	**47**	2,209	6.856	**80**	6,400	8.944
15	225	3.873	**48**	2,304	6.928	**81**	6,561	9.000
16	256	4.000	**49**	2,401	7.000	**82**	6,724	9.055
17	289	4.123	**50**	2,500	7.071	**83**	6,889	9.110
18	324	4.243	**51**	2,601	7.141	**84**	7,056	9.165
19	361	4.359	**52**	2,704	7.211	**85**	7,225	9.220
20	400	4.472	**53**	2,809	7.280	**86**	7,396	9.274
21	441	4.583	**54**	2,916	7.348	**87**	7,569	9.327
22	484	4.690	**55**	3,025	7.416	**88**	7,744	9.381
23	529	4.796	**56**	3,136	7.483	**89**	7,921	9.434
24	576	4.899	**57**	3,249	7.550	**90**	8,100	9.487
25	625	5.000	**58**	3,364	7.616	**91**	8,281	9.539
26	676	5.099	**59**	3,481	7.681	**92**	8,464	9.592
27	729	5.196	**60**	3,600	7.746	**93**	8,649	9.644
28	784	5.292	**61**	3,721	7.810	**94**	8,836	9.695
29	841	5.385	**62**	3,844	7.874	**95**	9,025	9.747
30	900	5.477	**63**	3,969	7.937	**96**	9,216	9.798
31	961	5.568	**64**	4,096	8.000	**97**	9,409	9.849
32	1,024	5.657	**65**	4,225	8.062	**98**	9,604	9.899
33	1,089	5.745	**66**	4,356	8.124	**99**	9,801	9.950

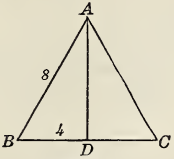
Square Root in Geometry

1. Find the side of a square equal in area to a rectangle whose length is 342 ft. and whose width is 195 ft.

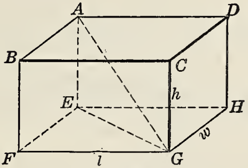
2. Find the diagonal of a square whose side is 10.

3. Find the diagonal of a rectangle whose sides are 12 and 18.

4. Find to three significant figures the side of a square whose area is 8 sq. ft.

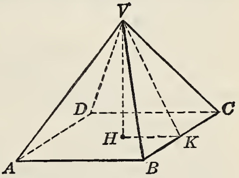


Ex. 5



Ex. 6

5. ABC is an equilateral triangle whose side is 8. What is the length of the altitude AD if it cuts the base in halves and makes a right triangle ABD?

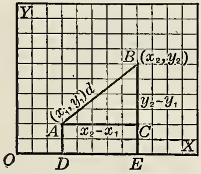


6. The diagonal of a box is given by the formula

$$d = \sqrt{l^2 + w^2 + h^2}$$

Find d when $l = 12$, $w = 8$, and $h = 14$.

7. In the square pyramid $VABCD$, the altitude VH makes a right angle with HK, and H is the center of the square. Find the slant height VK if the altitude $VH = 12$ and the sides of the square base are each 18.



How to find the distance between points on a graph. To find the distance from A to B, complete the right triangle ABC. Then $d^2 = AC^2 + BC^2$. But $AC = x_2 - x_1$ and $BC = y_2 - y_1$. So by the Pythagorean Theorem:

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Find the distance between these points:

8. (5, 6) and (2, 2)
9. (8, 5) and (6, 2)
10. (7, 6) and (– 5, 1)
11. (– 3, – 5) and (4, – 1)
12. (0, 5) and (0, 0)
13. (0, 0) and (– 3, – 4)
14. (2, 0) and (0, 2)
15. (– 7, 0) and (0, – 6)

Square Root Applied

1. A baseball diamond is a square 90 ft. on a side. Find the distance from home plate to second base.

2. Can Dorothy pack her umbrella which is 27 in. long in the bottom of her suitcase which is 23 in. long and 15 in. wide?

3. A vacant lot is 100 ft. by 80 ft. What is the length of a path diagonally across it?

4. A carpenter is making a gate 4 ft. wide and 3 ft. high. How long should he cut the diagonal piece used to strengthen the gate?

5. How long a wire will be needed to reach from the top of a pole 16 ft. high to a point on the ground 12 ft. from the foot of the pole?

6. A vacant corner lot is 120 ft. long and 50 ft. wide. What is the length of a straight path running diagonally across the lot joining two opposite corners? What distance is saved by taking the path?

7. A ship sails 30 mi. east and then 40 mi. north. How far is it then from its starting point?

Square Root in Science and Engineering

1. The diameter d of a pipe that will carry as much water as 2 pipes whose diameters are d_1 and d_2 is calculated from the formula $d = \sqrt{d_1{}^2 + d_2{}^2}$. Find to the nearest inch the diameter of a pipe that will supply 2 pipes whose diameters are 5 in. and 7 in. Will two 3-in. pipes carry as much water as a 6-in. pipe?

2. The pressure of the air on the wing of an airplane depends on the velocity of the airplane and is given by the formula $p = .0005\,v^2$. What value of v is needed to make p equal to 12?

3. The time it takes a pendulum to swing depends on its length and is given by the formula $t = \pi\sqrt{\frac{l}{32}}$. How long will it take a pendulum whose length is 42 in. to make a single swing?

4. When a stone is dropped from a height, the distance fallen at the end of t seconds is $d = 16\,t^2$. How long will it take for it to fall 800 ft.?

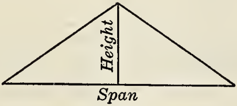
5. The velocity of water flowing under a head is $v = c\sqrt{64\,h}$.

What is the velocity to three significant figures when $c = .500$ and $h = 14.6$?

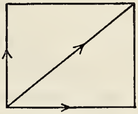
6. A carpenter must order rafters for the gable of a house. What length rafters will he need if the height of the gable is 12 ft. and the span is 20 ft.?

7. The distance between rivets in a steam boiler is found by means of the formula $d = \sqrt{\frac{(11\,p + 8\,r)(p + 8\,r)}{10}}$. Find d when $p = 3$ and $r = \frac{1}{2}$.

8. A boat travels 40 ft. across a river while the current carries it 30 ft. down-stream. Draw the path of the boat, and find the distance it has moved.



Ex. 6



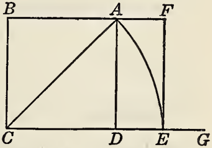
Ex. 10

9. A man walks at the rate of 5 mi. an hr. across the deck of a boat that is traveling 12 mi. an hr. What is his actual speed?

10. A force of 60 lbs. is pulling directly north and a force of 80 lbs. directly east. If the resultant force is represented by the diagonal of the rectangle whose sides are as many inches long as the forces are pounds, find the amount of this force.

Square Root in Art

Artists think that a picture is more beautiful if its length is $\sqrt{2}$ or $\sqrt{3}$ or $\sqrt{5}$ times as long as the width. These shapes are called ***root 2, root 3,*** and ***root 5 rectangles.***

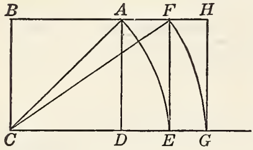


1. A landscape painter wishes to make a root 2 picture, so he starts with a square *ABCD* whose sides *AD* and *CD* are each 10 in. Next he takes *AC* as a radius and *C* as center and makes an arc cutting *CG* at *E*. Finally he draws the rectangle *FBCE* whose base *CE* equals the diagonal *CA*.

(*a*) Does this give him a root 2 rectangle? Explain.

(*b*) Find the length of CE to the nearest tenth. Is this length $\sqrt{2}$ times the width 10?

2. When an artist wants a root 3 rectangle, he first makes a root 2 rectangle $FBCE$ as explained above. Then he takes its diagonal CF as radius and C as center and draws the arc FG. If $FE = 1.00$ and $CE = 1.41$, find CF and CG to the nearest hundredth. Now find the square root of 3 to the nearest hundredth. Does CG equal the $\sqrt{3}$?



RADICAL EQUATIONS

Paul was repairing an old Grandfather's clock. He wanted to make the pendulum just long enough so it would beat seconds. In a book he found the formula $t = \pi\sqrt{\frac{l}{32}}$, where l is the length of the pendulum. An equation such as this containing a root sign is called a ***radical equation.*** Paul had to solve this equation for l. How could he do it? To get rid of a square root, he must perform its opposite. But what is the opposite of square root?

Here is Paul's work: $t = \pi\sqrt{\frac{l}{32}}$

Dividing by π: $\frac{t}{\pi} = \sqrt{\frac{l}{32}}$

Squaring both sides: $\frac{t^2}{\pi^2} = \frac{l}{32}$.

Multiplying by 32: $\frac{32\,t^2}{\pi^2} = l$

Substituting $t = 1$ and $\pi = 3.14$, he found $l = 3.25$ ft.

Rule. To get rid of a square-root sign:

1. Get the radical alone on one side of the equation.

2. Square both sides of the equation.

Thought Question

How would you get rid of a cube-root sign ($\sqrt[3]{\ }$)?

One peculiarity of radical equations is that sometimes they have no answer. Consequently, it is necessary to check, for even though you make no mistake in your work, your result may not be an answer.

Illustration. Solve: $\sqrt{x-3} = 4$
Squaring: $x - 3 = 16$
Transposing: $x = 19$
Check: $\sqrt{19-3} \mid 4$
$\sqrt{16} \mid 4$
$4 = 4$

Class Exercises

Solve and check:

1. $\sqrt{x} = 5$
2. $\sqrt{3x} = 6$
3. $\sqrt{x+4} = 3$
4. $\sqrt{x-2} = 4$
5. $\sqrt{2x+3} = 5$
6. $\sqrt{3x-2} = 7$
7. $\sqrt{\frac{1}{2}x} = 2$
8. $\sqrt{.4x} = 8$
9. $\sqrt[3]{x} = 4$
10. $\sqrt[3]{x+2} = 3$
11. $\sqrt{x} - 3 = 2$
12. $\sqrt{x} + 5 = 7$
13. $\sqrt{3x} + 4 = 1$
14. $\sqrt{x+4} - 5 = 0$
15. $\sqrt{x-3} + 2 = 0$
16. $\sqrt[3]{x-7} = 2$
17. $\sqrt{x} = 8 - \sqrt{x}$
18. $2\sqrt{x} - 9 = \sqrt{x}$

Optional Exercises

19. $\sqrt{2x+.15} = .5$
20. $2\sqrt{3x+.1} = 2$
21. $\sqrt{.7x-.17} = .5$
22. $8 - \sqrt{3x-1} = 5$
23. $3 + \sqrt[3]{x+4} = 5$
24. $7 - 3\sqrt{x+4} = 2 + 2\sqrt{x+4}$

In these formulas find the value of the unknown letter:

25. $t = \sqrt{\frac{2s}{g}}$ $t = 5$ and $g = 32$
26. $r = \sqrt{\frac{V}{\pi h}}$ $V = 942$, $\pi = 3.14$, and $r = 10$
27. $r = \sqrt[3]{\frac{3V}{4\pi}}$ $r = 7$ and $\pi = \frac{22}{7}$
28. $c = \sqrt{a^2 + b^2}$ $c = 13$ and $a = 5$

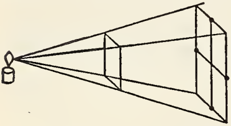
The Radical Equation in Science

1. The voltage needed to produce a certain amount of heat in an electric iron is $V = \sqrt{\frac{Hr}{t}}$. Find the amount of heat produced in an iron if $V = 110$ volts, $r = .05$ ohms, and $t = 60$ sec.

2. The radius of the pump on a fire engine necessary for pumping g gals. of water a sec. is $r = \sqrt{\frac{231\,g}{\pi LN}}$ where L is the length of the stroke and N is the number of strokes a sec. How many gallons a second will be pumped by an engine if $r = 7$, $L = 10$, and $N = 21$?

3. The diameter of a motor-boat engine's cylinders to produce a certain horse-power is $d = \sqrt{\frac{13\,h}{n}}$. What horse-power h has an engine if the diameter d is 10 and the number of cylinders n is 8?

4. A formula used in magnetism is $T = 2\,\pi\sqrt{\frac{I}{MH}}$. Find I when $T = 22$, $M = 100$, and $H = 240$.



5. The amount of illumination on your book from a light depends on the distance of the light from your book. If the distance becomes twice as great, the area the light spreads over becomes 4 times as great, and therefore the intensity of illumination becomes only $\frac{1}{4}$ as great.

(*a*) How many times as much light will fall on the pages of your book if you move up from a distance of 10 ft. from the light to a distance of 2 ft. from it?

(*b*) If $d = \sqrt{\frac{k}{I}}$, find I when $d = 5$ and $k = 50$.

6. The radius of a cylindrical vegetable can is given by the formula $r = \sqrt{\frac{V}{\pi h}}$. What value must r have if $h = 3$ in., so that V may be 44 cu. in.? So that V may be 30 cu. in.?

7. In the hydraulic press, used for exerting great pressure, the diameter D of the large cylinder is given by the formula

$D = d\sqrt{\frac{F}{f}}$. Find the force F if the force $f = 100$ lbs., the diameter of the small cylinder $d = 1$ in., and the diameter D of the large cylinder $= 20$ in.

8. A silo for storing feed for cattle is usually a cylinder standing on end. The radius of a silo 10 ft. high that will hold t tons of feed is $r = \sqrt{\frac{5\,t}{\pi}}$. If the silo has a radius of 6 ft., how many tons will it hold?

Review Exercises

1. Find the amount of $5840 at 5% compounded annually for 7 yrs.

2. Solve for r: $h = \frac{(w + w')gr^2}{12\,wv^2}$

3. The U.S. Marine rule for the diameter of rivets in building ships is $d = \sqrt{\frac{w}{.78 \times 12000\,n}}$, where n is the number of rivets and w is the weight supported. Solve for w.

4. Solve for x and y:

$$\begin{aligned} 3\,x - 5\,y &= 6 \\ 2\,x + 3\,y &= 23 \end{aligned}$$

5. At what price should a furniture dealer sell a chair that cost $18 so that he can make 40% of the selling price?

6. Find $\sqrt{2}$ to four significant figures.

7. Express 237,741 to three significant figures.

8. Factor:

(*a*) $3\,x + 6\,y$ (*c*) $27\,r^2w^3 - 18\,r^2w$

(*b*) $14\,n + 7$ (*d*) $1.5\,x + 4.5$

9. Stock selling for $45 a share pays $3 a share dividend. What per cent interest on the investment does it pay?

10. Find to the nearest hundredth a leg of a right triangle:

(*a*) If the hypotenuse is 16 and the other leg 13.

(*b*) If the hypotenuse is 20 and the other leg 14.

(*c*) If the hypotenuse is 70 and the other leg 47.

11. Solve for x:

(*a*) $2\sqrt{x-5} = 14$
(*b*) $\sqrt{2x-7} - 5 = 0$
(*c*) $\sqrt{x} + 4 = 0$
(*d*) $\sqrt[3]{x+2} - 3 = 0$

12. Find to three significant figures the square root of:

(*a*) 387
(*b*) 21.6
(*c*) 89540
(*d*) 7.194
(*e*) .005
(*f*) $\frac{3}{8}$

13. Express to two significant figures:

(*a*) 3.764
(*b*) .0356
(*c*) 1.085
(*d*) 2.99
(*e*) 8472
(*f*) 698.54

14. If c is the hypotenuse of a right triangle whose base and altitude are b and a, then $c = \sqrt{a^2 + b^2}$. In these exercises find the unknown side:

(*a*) $a = 5$, $c = 13$
(*b*) $b = 20$, $c = 25$
(*c*) $a = 8$, $b = 15$
(*d*) $a = 40$, $c = 41$
(*e*) $c = 83$, $a = 57$
(*f*) $a = 61$, $b = 54$
(*g*) $a = 8.5$, $c = 11.6$
(*h*) $b = .037$, $c = .042$

15. A ladder 20 ft. long has its base 11 ft. from the wall of a building. How far up the building will it reach?

16. A team of small boys wish to lay out a baseball diamond. As their third baseman cannot throw the ball across the regulation diamond, what length should they take for a side of their diamond so that the distance from third base to first base will be 80 ft.?

Chapter 12

RATIO AND PROPORTION

IN his will a man left his wife \$12,000 and his son \$8000, but when he died, it was found that his estate amounted to only \$15,000. How much should each receive?

The probate court decided that the estate should be divided in the ratio of 12,000 to 8000. What did it mean? What is the ratio of 12,000 to 8000? Can you divide \$15,000 in this ratio?

This is but one of the many problems that occur in life, in law, in business, in art, in chemistry, in home economics, in engineering, in music, in physics and mechanics, and in almost every line of human endeavor, in which a knowledge of ratio is helpful.

The *ratio* of two numbers is the quotient obtained by dividing the first by the second. For example, the ratio of 12 to 20 is $\frac{12}{20}$ or $\frac{3}{5}$. A ratio then is just a fraction. Its numerator and denominator are called the *terms* of the ratio.

The ratio $\frac{a}{b}$ is also written in the form $a:b$. In either form it is read "*a* is to *b*."

Since the ratio is a fraction, it may also be expressed as a decimal. For example, $3:8 = \frac{3}{8} = .375$.

Exercises

Find the ratio of:

1. 33 to 44
2. 840 to 930
3. 12 days to 30 days
4. 8 ft. to 18 ft.
5. \$600 to \$900
6. $\frac{1}{4}$ to $\frac{3}{4}$
7. $\frac{2}{3}$ to 1
8. $\frac{6}{7}$ to $\frac{2}{7}$

Express these ratios as decimals correct to three significant figures:

9. 5:6

10. 8:15

11. 48:90

12. $\frac{1}{5}:\frac{3}{5}$

13. 48:16

14. 13:13

Find these ratios to hundredths, and arrange them with the larger ratio first:

15. 5 to 8 or 2 to 3

16. 13:20 or 16:25

17. 256 to 288 or 288 to 320

18. $341\frac{1}{3}:426\frac{2}{3}$ or 256:320

19. Baseball records are expressed as the ratio of the number of games won to the number played. This ratio expressed to thousandths is called the per cent of the team, although it is not the ordinary use of the word "per cent." Find the ratio or "per cent" for each team in the American League, if the games won and lost in a certain season were as follows:

Team	*Won*	*Lost*	*Team*	*Won*	*Lost*
Detroit..........	93	58	Chicago.........	74	78
New York.......	89	60	Washington.....	67	86
Cleveland........	82	71	St. Louis........	65	87
Boston..........	78	75	Philadelphia.....	58	91

20. A basketball team won 7 games out of 12 played. If it should win its next two games, by how much would its percentage be increased?

21. A flag-pole 60 ft. high casts a shadow 45 ft. long. What is the ratio of the height of the pole to the length of its shadow?

22. In 1920 there were 95,000,000 white persons and 10,000,000 negroes in the United States. What was the ratio of the whites to the negroes?

23. In a single year the number of births in the United States was 1,600,000 and the number of deaths was 850,000. Find the ratio of the number of deaths to the number of births.

24. A baseball player's batting average is the ratio of the number of hits to the number of times at bat. It is expressed

Photo by Charles Phelps Cushing, R.I. Nesmith, N.Y.

A WORLD SERIES GAME

as a decimal to three significant figures. Find the batting averages of these National League players for a certain year, and arrange them in order, placing the highest of the averages first.

Name	*Team*	*At Bat*	*Hits*
Harnett	Chicago	413	142
Herman	Chicago	657	222
Lombardi	Cincinnati	333	114
Medwick	St. Louis	626	223
Terry	New York	596	203

25. A certain angle contains 75°. What is its ratio to its supplement?

26. A tax rate is the ratio of the amount of money to be raised to the assessed valuation of the town. Find to 3 significant figures the tax rate if the amount needed is \$87,400 and the assessed valuation of the town is \$2,460,000.

Ratio Problems

If two numbers are in the ratio $\frac{a}{b}$, we can represent them by ax and bx, for $\frac{ax}{bx} = \frac{a}{b}$. Two numbers in the ratio 3:7 are represented by $3x$ and $7x$.

Illustration: Find two numbers in the ratio 4:5 whose sum is 162.

Let $4x =$ the first number
$5x =$ the second number
$4x + 5x = 162$
$x = 18$
$4x = 72$
$5x = 90$

Check:
$72:90 = 4:5$
$72 + 90 = 162$

1. Find two numbers in the ratio 5:8 whose sum is 65.

2. Find two numbers in the ratio 7:4 whose difference is 51.

3. Separate 108 into two parts in the ratio 5:7.

4. Three numbers are to each other as 3:5:7, and their sum is 45. Find them.

5. Three numbers are to each other as 3:5:6, and the sum of the first two is 8 more than the third.

6. Find two supplementary angles in the ratio of 3:1.

7. Find the angles of a triangle if they are to each other as 3:7:8.

8. The length of a rectangle is to the width as 7:5, and the perimeter is 132. Find the dimensions.

9. Two numbers are in the ratio 6:5. If I add 4 to the smaller, they will be equal. Find the numbers.

10. Find the smallest angle of a triangle if the three angles are to each other as 1:2:2.

11. In a class of 35 pupils, the ratio of the number of boys to the number of girls is 4:3. How many of each are there?

12. Find two numbers whose ratio is 4 and whose sum is 40.

13. Two numbers are in the ratio 3:8. If 7 is added to the smaller, they are then in the ratio 2:3. Find the numbers.

14. In surveying, the Egyptians divided a rope into 3 sections that were to each other as 3:4:5. Into what lengths would they divide a rope 90 ft. long?

The Ratio Applied

1. An average family spends for food and clothing in the ratio 5 to 2. If the Jones family can afford to spend $840 a year for both items, how much should they spend on each?

2. Solder is an alloy of lead and tin in the ratio 2:1. How much of each metal is needed to make 10 lbs. of solder?

3. How much antimony is needed to make 100 lbs. of type metal, if lead and antimony are used in the ratio 4 to 1?

4. Mr. Wills left $3000 to be divided among his 3 children in the ratio 3:4:5. Find the share of each.

5. Common salt is composed of sodium and chlorine in the ratio 23: 35. How many grams of salt must be used to obtain 105 g. of chlorine?

6. A common iron ore contains iron and oxygen in the ratio 7:3. How many tons of ore must be reduced to produce 35 T. of iron?

7. An important ore of zinc contains zinc and sulphur in the ratio 2:1. How much zinc can be taken from 600 lbs. of this ore?

8. Two men went into partnership investing $4000 and $6000. During their first year they made a profit of $1200. If they divide this in the ratio of their investments, how much should each receive?

9. I pay $200 a yr. taxes on a piece of property worth $6000. What should another piece of property be worth if it is taxed $240 a yr.?

10. John Edwards bought a lot for $6000 and built a house on it costing $4000. During the depression the whole property depreciated to $7000. If he wishes to insure the house for its present value, how much insurance should he carry on it?

11. A house worth $3000 is insured for $2400 with the understanding that the insurance company will pay such part of the loss as the insurance is of the value of the property. If a fire damages the house to the extent of $800, how much should the company pay?

12. A building is insured for $3000 in one company and for $4000 in another. How much should each company pay if fire injures it to the amount of $2100?

13. A bank having deposits of \$1,000,000 and assets of \$600,000 failed. Mr. Adams has \$800 in his account. How much should he receive on his claim?

14. Concrete is made of cement, sand, and broken stone in the ratio 1:2:4. How much of each is needed to make 3500 lbs. of concrete?

15. In a poorer concrete the cement, sand, and stone are used in the ratio 1:3:6. Find the amount of each needed to make 1200 lbs. of this concrete.

16. In his will Robert Palmer left \$20,000 to his wife, \$9000 to his son, and \$10,000 to his daughter. At his death his estate amounted to \$65,000. How much should each receive?

17. Three partners invested \$8000, \$10,000, and \$15,000 in a business. If a profit of \$44,000 is to be divided in the ratio of their investments, how much should each receive?

PROPORTION

A *proportion* is a statement that two ratios are equal. For example, $\frac{8}{10} = \frac{12}{15}$ is a proportion. We read "8 is to 10 as 12 is to 15" which means that 8 is the same part of 10 that 12 is of 15. We can write $8:10 = 12:15$.

In a proportion the first and last terms are called *extremes* and the second and third terms are called *means.*

In the above equation 8 and 15 are the extremes, and 10 and 12 are the means. In the proportion $\frac{a}{b} = \frac{c}{d}$, the extremes are a and d, the means b and c.

HISTORICAL NOTE ON RATIO AND PROPORTION: Various forms of proportion were used by early writers, Euclid in particular using it in his geometry. Brahmagupta (628) uses a form of it called the rule of three. Most writers gave an arbitrary rule for solving problems, and it was not until about 1500 that a proportion was recognized as the equality of two ratios. Inverse proportion and compound proportion were also studied. Problems that we now solve by simple equations were solved by guessing an answer and then correcting it by proportion. Most of the problems were fanciful and absurd.

Since a proportion is simply a fractional equation, it can be solved just as any other fractional equation can.

Examine the proportion $8:10 = 12:15$. What is the product of the extremes? Of the means? Try several other proportions. Is the same relation true?

In a proportion, the product of the extremes equals the product of the means.

In the proportion $\frac{a}{b} = \frac{c}{d}$, the common denominator is bd. Multiplying by bd, we get $\frac{abd}{b} = \frac{cbd}{d}$ which reduces to $ad = bc$.

This property is very useful either in testing a proportion or in solving for an unknown term.

Illustration 1. Is $\frac{4}{7} = \frac{5}{9}$ a true proportion? The product of the extremes is $4 \times 9 = 36$. The product of the means is $7 \times 5 = 35$. Since the products are not equal, the proportion is not true.

Illustration 2. Find x if $\frac{x}{4} = \frac{5}{8}$. By the products of extremes and means, we have $8x = 20$. Therefore, $x = 2\frac{1}{2}$.

Class Exercises

1. Determine which of these proportions are true:

(*a*) $\frac{3}{5} = \frac{8}{13}$

(*b*) $\frac{7}{11} = \frac{14}{22}$

(*c*) $\frac{4.2}{5} = \frac{6.3}{7.5}$

2. Find the value of x in these proportions:

(*a*) $\frac{3}{x} = \frac{5}{8}$

(*b*) $\frac{2}{4} = \frac{5}{x}$

(*c*) $\frac{3}{5} = \frac{8}{x}$

(*d*) $\frac{x}{3} = \frac{16}{9}$

(*e*) $\frac{7\frac{1}{2}}{3} = \frac{x}{15}$

(*f*) $\frac{3}{x} = \frac{4}{7}$

(*g*) $\frac{x}{2.4} = \frac{5}{8}$

(*h*) $\frac{5}{4} = \frac{x}{8}$

(*i*) $\frac{2.7}{9} = \frac{3}{x}$

(*j*) $\frac{3}{5.5} = \frac{x}{33}$

(*k*) $\frac{3.4}{x} = \frac{1.7}{2.4}$

(*l*) $\frac{x}{3.2} = \frac{12}{12.8}$

(*m*) $\frac{4.2}{3.5} = \frac{2.4}{x}$ (*n*) $\frac{2\frac{1}{2}}{7\frac{1}{2}} = \frac{x}{4\frac{1}{2}}$ (*o*) $\frac{x}{.003} = \frac{50}{.06}$

3. Find the ratio $x:y$ if:

(*a*) $3x = 2y$ (*b*) $x = y$ (*c*) $5x - 4y = 0$

Optional Exercises

4. Solve for x:

(*a*) $\frac{x+2}{6} = \frac{5}{3}$ (*e*) $\frac{x}{5} = \frac{a}{b}$ (*i*) $\frac{x-a}{x-b} = \frac{c}{d}$

(*b*) $\frac{x-1}{4} = \frac{7}{2}$ (*f*) $\frac{x}{a} = \frac{m}{n}$ (*j*) $\frac{ax-1}{bx+2} = \frac{3}{4}$

(*c*) $\frac{x}{x+3} = \frac{3}{4}$ (*g*) $\frac{k}{x} = \frac{5}{n}$ (*k*) $\frac{x+a}{x+b} = \frac{r}{s}$

(*d*) $\frac{x}{4-x} = \frac{3}{7}$ (*h*) $\frac{a}{bx} = \frac{c}{d}$ (*l*) $\frac{mx+n}{px+q} = \frac{g}{h}$

Honor Work

5. Solve for x:

(*a*) $\frac{x+2}{x+3} = \frac{x-1}{x-2}$ (*c*) $\frac{ax+2}{bx+5} = \frac{ax+9}{bx-3}$

(*b*) $\frac{x+a}{x-b} = \frac{x-c}{x+d}$ (*d*) $\frac{ax+p}{bx+q} = \frac{ax+m}{bx+n}$

The Ratio in Business

1. Brown and Smith formed a partnership, Brown investing $7200 and Smith $9600. The first year their profits were $2800. If they divided in the ratio of their investments, how much did each receive?

2. A town must raise $22,000 by taxes. If the total valuation of the town is $1,000,000 and the value of Henry Dunn's property is $25,000, what tax should he pay?

3. A tax rate is the ratio of the amount of money to be raised to the total valuation of the town. If the valuation is $1,600,000 and the amount to be raised is $27,000, find the tax rate to five decimal places.

4. A house worth $12,000 is insured for $8000. If the insurance company pays in the ratio of the amount of insurance to 80% of the value of the house, how much should it pay for a fire loss of $1800?

5. Mr. Dean invested $9000 in stock and receives a dividend of $270. How much must Mr. Love have invested in the same stock if his dividend is $540?

6. As building lots on the same street generally have the same depth back from the street, their values are in the ratio of the number of front feet. If a lot with a frontage of 28 ft. sells for $400, what price should a lot with a frontage of 49 ft. bring?

7. A cylindrical tank, standing on end, is 8 ft. high and, when full, holds 680 gals. of gasoline. If the number of gallons of gasoline has the same ratio as the depth, how many gallons are in it when it is filled to a depth of 3 ft.?

8. A certain kind of concrete is made of cement, sand, and gravel in the ratio 2:7:11. How much of each will be required to make 400 lbs. of the mixture?

9. Mrs. Wells made up this family budget when Mr. Wells was getting a salary of $3000 a year: food, $800; clothing, $450; rent, $600; fuel and light, $200; other expenses, $450; and savings, $500. Now Mr. Wells salary has been raised to $3600. If she makes a new budget in the same ratio, how much will she allot to each item?

10. Burns, Sax, and Long form a partnership, investing respectively $9000, $7000, and $8000. (*a*) How should they divide a profit of $8400? (*b*) If first they take out a salary of $2000 each, how should they divide the remainder?

11. Three men go into partnership, investing 25%, 35%, and 40% of the capital needed. Each man is to receive a salary of $2000, and the remainder of the profit is to be divided in proportion to their investments. How should they divide a profit of $17,000?

12. Mr. Dixon had his house insured for $3000 in one fire-insurance company and for $2500 in another. What part should each pay if he has a fire loss of $2750?

13. In a certain life insurance company, a 20-payment life policy at age 22 costs $30.31 annually or $15.42 semi-annually. Is the rate proportional to the time? Which rate is cheaper?

Ratio on the Farm

1. A certain mixed feed is made from cracked corn and oats in the ratio of 3:2. How much of each should be used to make 800 lbs. of the feed?

Courtesy of International Harvester.

AN ELECTRIC CREAM SEPARATOR

The farmer pours the milk into the bowl at the top of the separator. It passes down over rapidly whirling discs which cause the cream to move to the center. Notice the cream coming from the spout on the left, and the remainder of the milk from that on the right.

2. A mash for chickens consists of 3 lbs. of corn meal to 8 lbs. of bran. How much must a farmer use to make 550 lbs. of the mash?

3. A cream separator breaks up whole milk into 3 qts. of cream to every 13 qts. of skimmed milk. How many quarts of whole milk are needed to produce 75 qts. of cream?

4. If a separator is set to furnish cream and milk in the ratio 2:9, how many quarts of cream will be obtained from 418 qts. of whole milk?

The Ratio in Sewing and Cooking

1. Directions for making ice cream say, " Use 4 parts of ice to 1 part of rock salt." How much of each should Gladys use to make 30 lbs. of the mixture?

2. A recipe for boiled rice, enough for 6 persons, is: 3 qts. of boiling water, 1 c. rice, and 2 t. of salt. Change the recipe to make enough for 10 persons.

3. Blanc mange is made from 2 c. of milk, $\frac{1}{4}$ c. corn starch, $\frac{1}{4}$ c. sugar, with a little vanilla, nutmeg, and salt. Disregarding these last ingredients, how many ounces of corn starch are needed for making 2 lbs. of blanc mange?

4. A food expert analyzed 2 lbs. of oatmeal and found that it contained $2\frac{1}{2}$ oz. water, 5 oz. of protein, 21 oz. of carbohydrates, and $3\frac{1}{2}$ oz. of other materials. Find the per cent of each ingredient.

5. To make fruit ice cream, use 2 c. of fruit juice, 2 c. of sugar, and 4 c. of cream. How many pounds of each will be needed to make 3 lbs. of ice cream?

6. Margaret is making Parker House rolls. The recipe calls for 4 c. flour, 1 t. salt, 6 t. baking powder, 2 T. shortening, and $1\frac{1}{2}$ c. milk. By mistake she put in 5 c. of flour. What amounts of the other ingredients should she take to make the recipe right?

7. Hazel's mother finds that the pattern she has been using will make a dress too small for Hazel now. One section of the pattern is 12 in. long and 4 in. wide. If she must make this section $\frac{1}{2}$ in. wider, how much should she lengthen it so as to keep the dress the same shape as that called for in the pattern?

8. This recipe for cream of tomato soup is enough for 7 people.

2 c. stewed tomatoes
$\frac{1}{8}$ t. baking powder
1 qt. milk
$\frac{1}{6}$ t. pepper
$\frac{1}{3}$ c. flour
$\frac{1}{4}$ c. butter
2 t. salt

Determine the approximate amount of each ingredient to make enough for 5 people.

9. A recipe for penoche says: Take 2 c. of brown sugar, $\frac{3}{4}$ c. milk, 2 c. chopped nuts, and a little butter and vanilla. Estimating the last two together as $\frac{1}{4}$ c., how many pounds of brown sugar will be needed to make 5 lbs. of penoche?

10. Here is a recipe for plain cookies:

1 c. sugar
1 T. shortening
1 egg
$\frac{1}{2}$ c. milk
1 t. baking powder
3 c. flour

If this recipe furnished enough cookies for 4 children, make a recipe that will supply 10 children.

11. Mrs. Hamilton crochets doilies 1 ft. in diameter which she sells for $1 each. A customer wants to know what price she would charge to make her a doily 20 in. in diameter. If the cost is proportional to the square of the diameter, what price should she ask?

The Ratio in Science and Engineering

1. Mr. Barnum finds that 3 T. of coal last 40 days. How much coal will he need for 90 days more?

2. Emily saw the lightning strike a tree 2300 ft. away from her, and she heard the crash 2 sec. later. She then timed another flash and found that the thunder came 7 sec. later. How far away was this lightning?

3. The pressure exerted by the water on a dam is proportional to the depth of the water above it. At a depth of 7 ft. the pressure on the dam is found to be 434 lbs. What is the pressure at a depth of 16 ft.?

4. A mining engineer finds that the amount of iron in an ore is to the amount of waste material as 4:7. How much of the ore will be needed to produce 20 T. of iron?

5. On a hydraulic jack it is necessary to use a force of 80 lbs. to raise a weight of 1440 lbs. What force will be needed to raise a weight of 3600 lbs.?

6. A water tank in the basement of a building can safely stand a pressure of 50 lbs. a sq. in. When water rises to a height of 34 ft. it causes a pressure of 15 lbs. a sq. in. on the tank. To how high a building can the tank safely supply water?

7. If five 50-watt electric lights use $\frac{1}{4}$ kw. of electricity an hr., how much will twelve 50-watt lights use?

8. A steel cable with a diameter of 3 cm. has a tensile strength of 90,000 lbs. If the strength is proportional to the square of the diameter, what size cable is needed to support 160,000 lbs.?

The Ratio in Chemistry and Medicine (Optional)

Nearly all of the substances you see around you—the wood of your desk, the clothing you wear, the food you eat, the medicines you take, and the air you breathe—are made up of a few simple elements such as carbon, hydrogen, nitrogen, etc. The chemist studies these elements and from them makes new substances that were never known to man before.

Here are a few of these elements with the letters that the chemist uses to stand for them, and with the weight of an atom of each.

Element	*Symbol*	*Atomic Weight*
Carbon	C	12
Hydrogen	H	1
Iodine	I	126
Nitrogen	N	14
Oxygen	O	16
Phosphorus	P	31
Potassium	K	39
Sulphur	S	32

In writing his formulas, the chemist leaves out plus signs, but we shall not do this because it would be too confusing. He also writes the multiplier after the letter instead of before it as we ordinarily do. For example, his formula for soap is K C 16 H 31 O. This means that soap contains 1 part of potassium (potash), 16 of carbon, 31 of hydrogen, and 1 of oxygen.

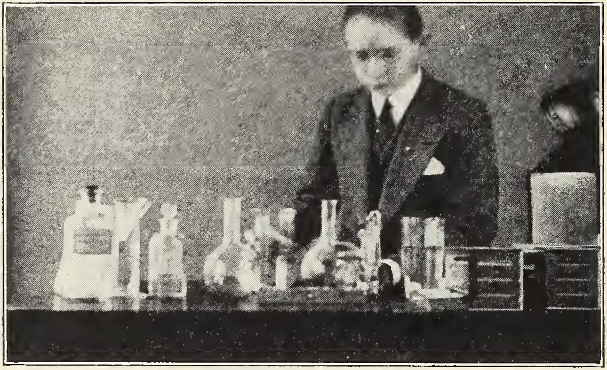
Atomic weight is simply a ratio for comparing the weights of elements. The atomic weight of hydrogen, the lightest element, is called 1. Then the atomic weight of oxygen, 16, means simply that an atom of oxygen is 16 times as heavy as an atom of hydrogen.

How to use chemical formulas.

Illustration: The farmer needs nitrogen in the soil to make plants grow, so he buys 400 lbs. of nitrate of potash and spreads it on his land. How much nitrogen does this give him? Nitrate of potash is $K + N + O\,3$.

Nitrate of potash $= K + N + O\,3 = 39 + 14 + 16 \times 3 = 101$

Nitrogen $= 14$.



CHEMISTRY AT HOME

This Boy Scout is making tests with his home chemistry outfit. With an outfit like this, any boy or girl can perform many interesting experiments.

	Atomic Weight	*No. of Lbs.*
Nitrate of potash	101	400
Nitrogen........	14	x

$$\frac{101}{14} = \frac{400}{x}$$
$$x = 55\tfrac{1}{2} \text{ lbs.}$$

1. Water is composed of oxygen and hydrogen. How many pounds of water would be needed to supply 6 lbs. of oxygen?

Formula for water: $H\,2 + O$

2. Ammonia contains nitrogen, hydrogen, and oxygen. How many pounds of nitrogen are there in 700 lbs. of ammonia?

Formula for ammonia: $N + H\,5 + O$

3. Gas from burning sulphur is often used for fumigating. To fumigate a certain room requires 5 lbs. of the gas. How many pounds of sulphur should be burned?

Formula for sulphur gas: $S + O\,2$

4. A chemist wants to prepare 7 lbs. of iodine from iodide of potash. How many pounds of it should he order?

Formula for iodide of potash: K + I

5. The Gold Match Company ordered 6200 lbs. of phosphorus for making matches. How many pounds of phosphate of potash will the chemical company use in making it?

Formula for phosphate of potash: K 3 + P + O 4

6. An important acid used in making glass and soap is called sulphuric acid because it is made from sulphur. How many pounds of sulphur are used in making 980 lbs. of this acid?

Formula for sulphuric acid: H 2 + S + O 4

7. Hydrogen, a gas used for filling airships because of its lightness, can be obtained from an acid. How many tons of sulphuric acid would be needed to furnish 2 T of hydrogen gas?

The Ratio in Art (Optional)

Artists find that the most beautiful proportions for a picture are those of the root 2, root 3, and root 5 rectangles. The ratio of the length to the width of these rectangles is as follows:

	Ratio of Length to Width
Root 2 rectangle	$\sqrt{2}:1$ or 141:100
Root 3 rectangle	$\sqrt{3}:1$ or 173:100
Root 5 rectangle	$\sqrt{5}:1$ or 224:100

1. How wide should a root 2 rectangle be if it is 18 in. long?

2. What length should a root 3 rectangle be if its width is 20 in.?

3. To fit the page of a book a picture must be reduced in size so that its length becomes $3\frac{1}{2}$ in. To what will the width be reduced if the picture is a root 5 rectangle?

4. The canvas that Rosemary is painting is 30 in. wide. What length should she use if she wants the picture to be a root 2 rectangle?

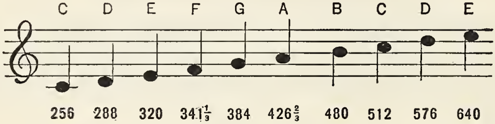
5. An artist has a canvas 20 in. wide and of proper length for a root 5 rectangle. If he decides to change to a root 3 rectangle, how many inches should he cut from the length?

6. Show that the two definitions of a root 5 rectangle given in the table are consistent; that is, that if the ratio of the length to the width is $\sqrt{5}$, then for a rectangle whose width is 100, the length should be 224.

7. A photograph is $7\frac{1}{2}$ in. long and 5 in. wide. If a copy is made from it $3\frac{3}{4}$ in. long, what will be the width of this copy?

The Ratio in Music (Optional)

When you strike a key on the piano, a little hammer hits a string. The string then makes the tone by vibrating back and forth very rapidly. The pitch of the tone depends on the number of times a second the string vibrates. Below is given the scale in the key of C with the numbers of vibrations a second that produce those tones.



Three or more notes played together produce a chord. This chord will be pleasant or unpleasant depending on the ratio of the vibration rates of the notes played. If the ratio is reduced to its lowest terms, the smaller the numbers the terms of the ratio reduce to, the more pleasant the chord will sound. This applies to the natural scale and not to the modified scale of which you will learn later. In an interval containing only two notes, you may consider the sound a pleasant one if the terms of the ratio are both smaller than 8.

Illustrations. The octave C:C = 256:512 = 1:2 is the most harmonious interval

$\frac{C}{G} = \frac{256}{384} = \frac{2}{3}$ is more pleasant than $\frac{C}{D} = \frac{256}{288} = \frac{8}{9}$.

Determine which of these are harmonious:

1. D and G
2. E and F
3. G and B
4. C and F
5. B and C
6. A and B
7. F and A
8. A and C
9. G and A

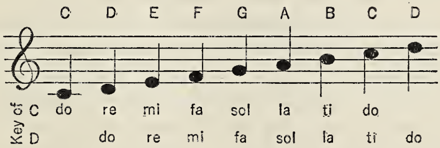
Which is the more harmonious:

10. C and G or C and F?
11. C and E or E and A?
12. C and E or F and A?
13. C and G or G and D?

Are these chords harmonious:

14. C, E, G, and C?
15. D, G, and B?
16. F, A, and C?

The modified scale. The numbers of vibrations given above are those of the natural scale. The scale actually in use on the piano differs a little from this and is called the modified scale. We shall now discover why this scale was needed.


C D E F G A B C D
Key of C do re mi fa sol la ti do
Key of D do re mi fa sol la ti do

Investigating the modified scale.

1. Find the ratios in the key of C for do: re, for do: mi, for do: fa.

2. Find the ratios in the key of D for do: re, for do: mi, for do: fa.

3. In the natural scale, is the do: re ratio the same in the key of D as in the key of C? Is the do: mi ratio the same? The do: fa? Do they differ much? Do you think that this difference would matter?

Now in music, when we transpose from the key of C to that of D, we do not wish to change the tune. We wish to change the pitch only. But in exercises 1 and 2 above we noticed that the ratios in the key of D were not the same as those in the key of C. This would give an entirely different effect and would change both the chords and the tune. To avoid this, the pitch of D is slightly changed on the piano. The chords then are not quite perfect for either C or D, but they are only slightly off and have the advantage of being the same for both keys. Similarly the other notes of the scale are shifted a little in pitch so that they will give the same effect for one key as for any other.

Why we have black keys on the piano. If we transpose to the key of E, E becomes do and F ought to become re. Determine what the do: re ratio is in this key. Is this ratio nearly the same as the do: re ratio in the key of C? Do you think that a slight shift of F would make the two ratios equal?

What would be the vibration rate of F if the E: F ratio were made equal to the C: D ratio?

In answering these questions, you should discover that the E: F ratio differs greatly from the C: D ratio. The difference is so great that if we attempted to shift F enough to make them alike, we would destroy the interval. To overcome this difficulty, F was left where it was, but an extra note was added to the piano scale between F and G and called F♯ (F sharp). The ratio E: F♯ was made equal to the ratio C: D. For other keys it was necessary to add other extra notes until five in all were added.

Because of this modified scale, certain combinations of notes, particularly in minor chords, have been made harmonious, although in the natural scale the terms of their ratios do not reduce to small numbers. For example, D: F, which equalled 27: 32, has been transformed into the same interval as E: G, which was 5: 6. Note that the ratio 27: 32 and 5: 6 differ only slightly.

Exercises

In any natural key in music, the tones should be such that the rates of vibration of its do : mi : sol : do are as 4 : 5: 6 : 8.

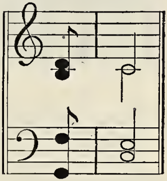
1. If an instrument is tuned to such a pitch that middle C vibrates 240 times a sec., find the vibration rates of the E, G, and C above this C.

2. In the key of G, G is do, and D is sol. Using the value of G just found, find the vibration rate of the D on the third line of the treble staff.

3. Two Boy Scouts have bugles tuned to the same pitch. If the only notes that can be gotten out of the bugles are sol, do, mi, sol, can the boys produce a discord? What notes should they play to get the most nearly perfect harmony?

4. What note other than the octave will sound the most harmonious with C? With F? With G? If you have a musical instrument convenient, check your results by trial.

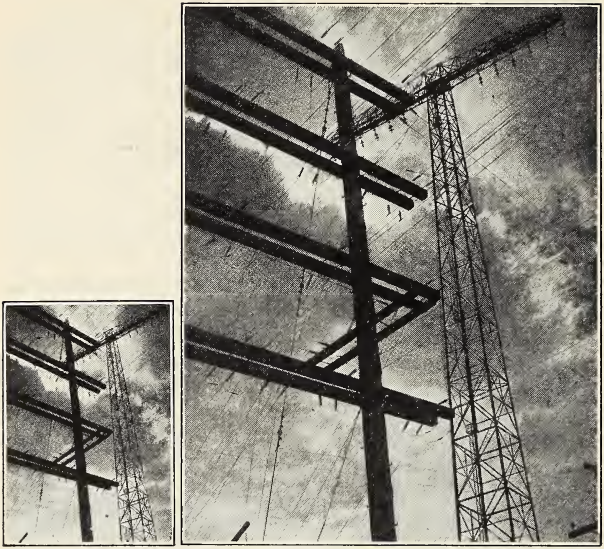
5. In writing music, it is customary to make the last chord very harmonious so as to leave a pleasant impression, and to make the chord just before it rather inharmonious so as to bring out the contrast. Determine the ratios for the last two chords of "Annie Laurie" as given here, and compare the harmony produced by them.



6. An instrument maker wishes to make a low-pitched instrument whose key-note, do, vibrates 48 times a sec. At what rates must he make the other notes of the scale—re, mi, fa, sol, la, ti, do—vibrate so the ratios will be the same as those for the natural key of C?

THE RATIO IN GEOMETRY: SIMILAR FIGURES

Any two figures that have the same shape are ***similar.*** Examine the picture and its enlargement. Do you think that they are similar? Does each length in one have the same ratio to the corresponding length in the other that any other two corresponding lengths have? How do the



Courtesy of Radio Corporation of America.

SIMILAR FIGURES

Does enlarging the picture change the size of angles? The length of lines? Do you think that the lengths in one picture are proportional to those in the other picture? The picture shows a radio transmission tower.

angles made by corresponding lines in the two pictures compare?

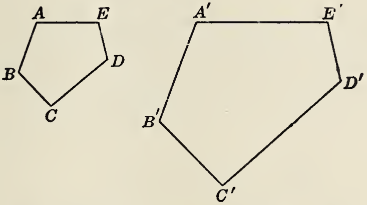
Two figures are ***similar*** if corresponding lines are proportional and corresponding angles are equal.

Similar figures have many uses. A map of a state is similar to the state; a plan of a house is similar to the floor section of the house; a model or statue is similar to the object modeled, etc.

Indirect measurement, such as finding a distance too great to be measured directly or to an inaccessible point, depends on the fact that corresponding lines in similar figures are proportional.

Exercises

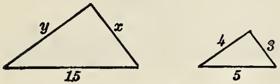
1. Here are two similar polygons. If $A'B'$ is twice as long as AB, how does $B'C'$ compare with BC? If BC is 8 in. long, how long is $B'C'$?



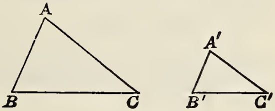
2. $ABCDE$ and $A'B'C'D'E'$ are similar. $AB = 4$, $BC = 3$, $CD = 6$, $DE = 2$, and $EA = 5$. Find the lengths of $B'C'$, $C'D'$, $D'E'$, and $E'A'$ if:

(*a*) $A'B' = 8$
(*b*) $A'B' = 12$
(*c*) $A'B' = 4$
(*d*) $B'C' = 90$

3. A picture is 3 in. long by 2 in. wide. If it is enlarged so that the length is $7\frac{1}{2}$ in., what is the width?



Ex. 4



Ex. 5

4. The base of a triangle is 5, and its other sides are 3 and 4. Find the sides of a similar triangle whose base is 15.

5. Triangle ABC is similar to triangle $A'B'C'$. Find the lengths of the sides not given.

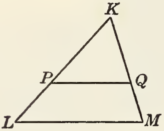
	AB	AC	BC	$A'B'$	$A'C'$	$B'C'$
(*a*)	8	14	10	12		
(*b*)	3			21	18	24
(*c*)		9	$10\frac{1}{2}$	5	15	
(*d*)	5.4	6.6	5.1			8.5

6. When a line is parallel to one side of a triangle, it cuts the other two sides into proportional segments. If PQ is parallel to LM, then:

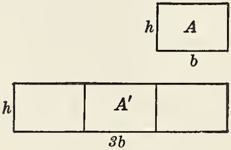
$$\frac{KP}{PL} = \frac{KQ}{QM}$$

If $KP:PL = 4:3$, what is the ratio $KQ:QM$?

7. (*a*) If $KP = 8$, $PL = 4$, and $KQ = 10$, what is the length of QM?



Ex. 6



Ex. 8

(*b*) If $KP = 12$, $PL = 8$, and $KM = 15$, what is the length of KQ?

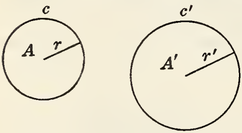
8. If the base of one rectangle is 3 times that of another rectangle while their altitudes are equal, how do their areas compare?

9. Calling the areas A and A', write a proportion.

Rectangles having equal altitudes are to each other as their bases.

10. The area of a rectangle is 40, and its base is 7. Another rectangle, having the same altitude, has a base of 21. What is its area?

11. You learned that the circumference of a circle is $c = 2\pi r$. The circumference of a second circle is $c' = 2\pi r'$. Obtain a proportion by dividing one equation by the other.



Circumferences are to each other as their radii.

12. The circumference of one circle is 5 times that of another circle. How do their radii compare?

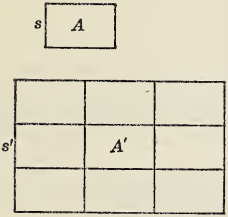
13. Find c' if $c = 25$, $r = 8$, and $r' = 4$.

14. Two rectangles A and A' have the same shape, but each side of A' is 3 times as long as the corresponding side of A. How many times as large as A is A'?

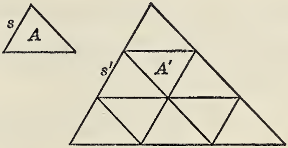
15. If A and A' are similar, how many times as large as A is A' if (*a*) $s' = 5\,s$? (*b*) $s' = 6\,s$?

Areas of similar rectangles are to each other as the squares of corresponding sides.

$$A : A' = s^2 : s'^2$$



Exs. 14, 15, 16



Exs. 17, 18

16. If A and A' are similar and s and s' are corresponding sides, find A':

(*a*) $A = 36$, $s = 2$, and $s' = 5$
(*b*) $A = 75$, $s = 5$, and $s' = 3$
(*c*) $A = 380$, $s = 3$, and $s' = 4.5$

17. A and A' are similar triangles. If s' is 3 times as long as s, how many times as large as A is A'?

18. In similar triangles A and A', how many times as large as A is A', if

(*a*) $s' = 2\,s$? (*b*) $s' = 5\,s$? (*c*) $s' = 4\,s$?

If a triangle grows larger without changing its shape, which increases more rapidly, the sides or the area?

HISTORICAL NOTE: Thales of Miletus (640–546 B.C.), a Greek philosopher and scientist, was probably the first to use similar triangles for measuring the height of a pyramid. His prediction of the time of a total eclipse of the sun in 585 B.C. made him famous, but his real claim to fame was the introduction of demonstrative geometry.

Areas of similar triangles are to each other as the squares of corresponding sides.

$$A:A' = s^2:s'^2$$

19. By what number is the area of a triangle multiplied when each side is multiplied by (*a*) 3? (*b*) 6? (*c*) n?

20. If $\frac{s}{s'} = \frac{4}{9}$, what is the ratio $\frac{A}{A'}$?

21. If $\frac{A}{A'} = \frac{4}{9}$, what is the ratio $\frac{s}{s'}$?

22. If A and A' are similar triangles and s and s' are corresponding sides, find A' when:

(*a*) $A = 12$, $s = 2$, and $s' = 3$
(*b*) $A = 9000$, $s = 30$, and $s' = 100$
(*c*) $A = 50$, $s = 5$, and $s' = 3$

23. A triangle is growing larger. How many times as great has its area become when its sides have each become (*a*) twice as large? (*b*) 3 times as large?

24. The area of a circle is given by the formula $A = \pi r^2$. What is the area of a circle whose radius is (*a*) 1? (*b*) 2? (*c*) 3?

25. By what number is the area of a circle multiplied when its radius becomes (*a*) twice as great? (*b*) 3 times as great? (*c*) n times as great?

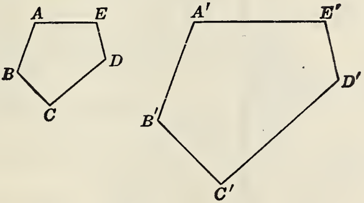
Areas of circles are to each other as the squares of their radii or as the squares of their diameters.

$$A:A' = r^2:r'^2$$
$$A:A' = d^2:d'^2$$

26. The area of a circle is 100. Find the area of a circle whose radius is 3 times as great.

27. If the area of a circle is 28.26 when $r = 3$, what is the area when r increases to 5?

28. We learned that the areas of similar rectangles vary as what? The areas of similar triangles vary as what? The areas of circles vary as what? Then if these two polygons K and K' were similar and if corresponding sides $s:s' = 2:3$, what would you expect the ratio $K:K'$ to be?



If we select polygons regular enough so that we can find their areas, or if we cut them into triangles and find their areas by taking the sum of the areas of the triangles, we find that:

The areas of similar polygons are to each other as the squares of corresponding sides.

$$\frac{K}{K'} = \frac{s^2}{s'^2}$$

29. If $K = 200$, $s = 10$, and $s' = 30$, find K'.

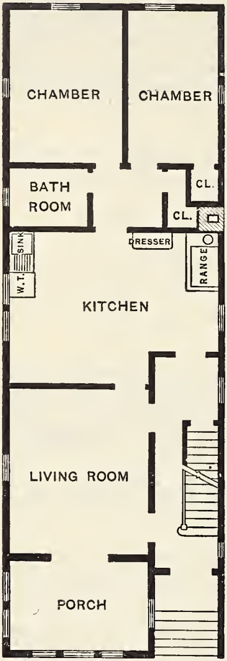
30. If $K = 98$, $s = 7$, and $s' = 5$, find K'.

31. If a polygon grows larger until its sides are 4 times as long as before, by what number is the area multiplied?

32. Sometimes the area of an irregular figure is determined by comparing its weight with that of a known area having the same thickness. If the weights of K and K' are in the ratio 16:25, what is the ratio of their sides?

NOTE: This fact about the areas of similar polygons has many important bearings on physics and astronomy. It is due to the fact that the areas of similar polygons vary as the squares of corresponding sides that light, sound, magnetism, gravitation, and electric forces vary inversely as the squares of distances.

The Ratio in Planning the Home



The plan shown in the diagram is that of a building 50 ft. long. By making measurements on the plan, answer these questions:

1. What is the width of the building?
2. Find the dimensions of the living-room and of the kitchen.
3. What is the width of the doorway between the living-room and porch?
4. What width is the hall beside the stairway?
5. Is there room enough in the living-room against the wall between it and the kitchen for a piano, the length of a piano being $4\frac{1}{2}$ ft.?
6. If a single width of carpet is to be laid the full length of the hall, how many yards will be required?

The Ratio in Life Insurance

To compute the amount of premium you should pay for a policy, a life insurance company must know how long you are likely to live. To determine this, they find how many out of a very large number of people will be alive a certain number of years from now.

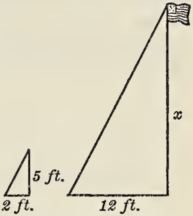
Here is a table showing how many, out of 1000 who reach the age of 10 yrs., will be alive at the ages given.

Age	*Number alive*
10	1000
30	890
50	718
70	382
90	16

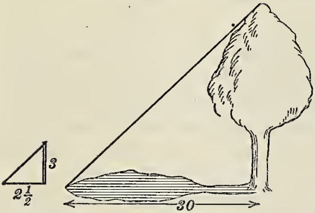
1. Out of 2000 at age 10, how many will be alive at 30? At 50? At 90?
2. Out of 44,500 at 30, how many will live to 70? To 90?

Ratio for the Boy or Girl Scout

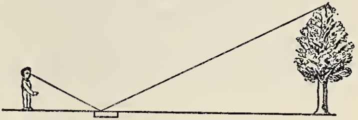
1. Here are a flag-pole and a post, both vertical; that is, if they stand on level ground, they make right angles with the ground. The sun causes them to throw shadows on the ground. Are the sun's rays parallel? How many equal angles can you find? Are the triangles similar? Can you make a proportion from which you can find x, the height of the flagpole?



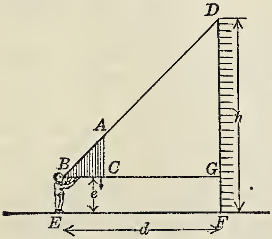
2. To find the height of a tree, some Girl Scouts measured the length of its shadow and then the shadow of the yardstick. If the tree cast a shadow 30 ft. long when the yardstick cast one $2\frac{1}{2}$ ft. long, how high is the tree?



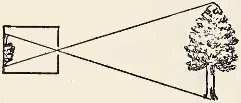
3. A vertical 6-ft. pole casts a shadow 5 ft. long at the same time that a building casts one 30 ft. long. How high is the building?



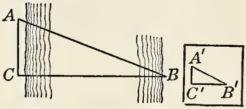
4. A Boy Scout found the height of a tree as follows: He placed a mirror horizontally on the ground and walked away from the tree until he could see the top of the tree in the mirror. If the mirror was 20 ft. from the tree and 4 ft. from the boy whose eye was 5 ft. from the ground, what was the height of the tree?



5. To measure the height of a building, make a right isosceles triangle ABC, and suspend a weight from a corner, A. Hold the triangle at the level of the eye so that the edge AC will be vertical (it will be vertical when the plumb-line hangs parallel to the edge) and walk toward or from the building until the highest point D is in line with B and A. Let e be the height of the eye from the ground, d the distance of the observer from the building, and h the height of the building. Then $h = d + e$. What is the height of a building if a girl whose eye is $4\frac{1}{2}$ ft. from the ground finds that she is 18 ft. from the building?



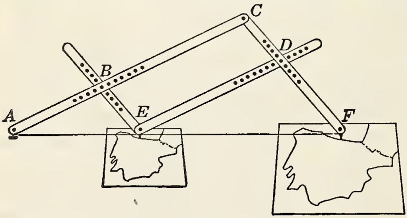
Ex. 6



Ex. 7

6. In a camera the size of a picture is to its distance from the lens of the camera as the size of the object is to its distance from the lens. If the camera is 6 in. deep and is held 10 ft. from a bush 4 ft. tall, what height will the picture have? If it is desired that the picture be 4 in. high, how far from the bush should the camera be held?

7. Robert wanted to find the distance across a river from C to B, so he measured the angles A and C with his protractor and drew a triangle on paper having angles A' and C' equal to those angles. He found AC to be 100 ft. If $A'C'$ is 2 in. and $C'B'$ is 5 in., what is the distance CB?



8. The pantograph is an instrument for enlarging drawings. If a drawing is traced by point E, a pencil at F will draw the enlarged figure. The ratio of the lines of the enlarged figure at F to those of the figure at E will equal the ratio of AC to AB. The Girl Scouts have a pantograph in which AC is 18 in. long. How long should they make AB so that the copy will be 3 times the original?

The Ratio in Geography

Maps and plans are similar to the objects of which they are maps or plans. Consequently all corresponding lines are proportional.

1. A map of the United States is drawn to the scale 1:12,000,000. Explain what this means, and tell what distance 1 in. represents.

2. How far is it from New York to Cleveland if the distance on this map is 2 in.

3. On this map the distance from Seattle to San Francisco is $3\frac{1}{2}$ in. What is the distance in miles?

4. On a map of Missouri, $1\frac{1}{2}$ in. represent 50 mi. Express the scale of this map as 1 to some number.

5. On this map of Missouri the distance from St. Louis to Jefferson City is 3 in. What is the distance in miles?

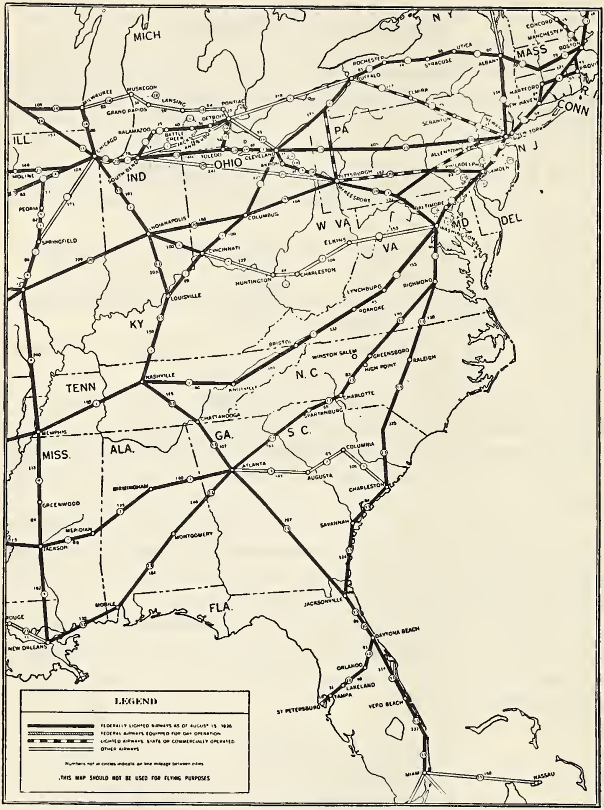
6. On a map of the United States, 1 ft. represents 1000 mi. If the area of the map is 3 sq. ft., what is the area of the United States?

7. A geography publisher wishes to illustrate by areas of similar figures the comparative amounts of wheat produced in different countries. If the ratio for two countries is 4:1, what ratio should he use for the sides of his figures?

8. On page 292 is an airways map of the eastern part of the United States. The air-line distance from New York to Chicago is about 700 mi. By measuring distances on the map, find the air-line distance between:

(*a*) Atlanta and Miami
(*b*) Memphis and New Orleans
(*c*) Buffalo and Chicago
(*d*) Boston and Detroit
(*e*) Cleveland and Washington
(*f*) Philadelphia and Pittsburgh

How things change together. Most things with which we have to deal in life change—your weight, your age, the number of pounds you can lift, the rate you can run, the


MICH
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GA.
ALA.
MISS.
FLA.
CONCORD
MANCHESTER
BOSTON
PROV
HARTFORD
NEW HAVEN
NEW YORK
ROCHESTER
BUFFALO
SYRACUSE
UTICA
ALBANY
ELMIRA
SCRANTON
ALLENTOWN
PHILADELPHIA
CAMDEN
PITTSBURGH
McKEESPORT
BALTIMORE
WASHINGTON
MILWAUKEE
MUSKEGON
GRAND RAPIDS
LANSING
PONTIAC
DETROIT
KALAMAZOO
BATTLE CREEK
JACKSON
ANN ARBOR
CHICAGO
SOUTH BEND
TOLEDO
CLEVELAND
AKRON
MOLINE
PEORIA
SPRINGFIELD
INDIANAPOLIS
COLUMBUS
CINCINNATI
HUNTINGTON
CHARLESTON
ELKINS
LOUISVILLE
LYNCHBURG
ROANOKE
RICHMOND
BRISTOL
WINSTON SALEM
GREENSBORO
HIGH POINT
RALEIGH
NASHVILLE
KNOXVILLE
CHARLOTTE
SPARTANBURG
CHATTANOOGA
COLUMBIA
MEMPHIS
ATLANTA
AUGUSTA
CHARLESTON
BIRMINGHAM
GREENWOOD
SAVANNAH
MERIDIAN
JACKSON
MONTGOMERY
MOBILE
JACKSONVILLE
ROUGE
NEW ORLEANS
DAYTONA BEACH
ORLANDO
LAKELAND
TAMPA
ST PETERSBURG
VERO BEACH
MIAMI
NASSAU
LEGEND
FEDERALLY LIGHTED AIRWAYS AS OF AUGUST 15 1930
FEDERAL AIRWAYS EQUIPPED FOR DAY OPERATION
LIGHTED AIRWAYS STATE OR COMMERCIALLY OPERATED
OTHER AIRWAYS
Numbers not in circles indicate air line mileage between cities
THIS MAP SHOULD NOT BE USED FOR FLYING PURPOSES

AIRWAY MAP OF EASTERN UNITED STATES

weather, the temperature, the value of your automobile, all change. And usually a change in one thing produces a change in another. A change in your age is usually accompanied by a change in your weight and your strength. A change in the direction of the wind is accompanied by a change in the weather. And it is true in mathematics and science generally that a change in one quantity produces a change in others. If a car is traveling at a constant speed, the distance it will go depends on the time it travels. The volume of a cube depends on the length of an edge. The length of a circular track depends on its radius.

A quantity that changes is called a ***variable,*** one that does not change is called a ***constant.***

Exercises

Name as many as you can of the principal variables on which these depend:

1. The amount of interest that I receive on an investment
2. The number of dozen oranges I can buy for $1
3. The cost of sending a parcel post package
4. The cost of painting my house
5. The distance a train can travel
6. The time a stone, that I throw, will remain in the air
7. The area of a rectangle
8. The number of pounds of grain a bin will hold
9. The number of feet of fence needed to enclose a rectangular lot
10. The length of the shadow cast by a post
11. The weather
12. The amount of money a salesman will earn
13. The number of rolls of wall paper needed for a room
14. The mark you get in algebra
15. The amount of money you will have when you are 30
16. The number of tons of coal needed to heat a building
17. The price of a pair of shoes
18. The amount of money your parents pay for electricity

How to make a formula from a table. The change in one quantity produced by a change in another can often be measured. Then the relation of the one variable to the other can be expressed, either as a rule, a formula, a graph, or a table. We have already made formulas from rules, and tables and graphs from formulas. It is not always possible to reverse this and make a formula from a table or graph, but when a formula can be made from a table, as in science, it is very often valuable.

Illustration. Make a formula from which this table might have been obtained. The cost c of sending a package weighing p lbs. by parcel post to a certain zone is given in this table:

p	1	2	3	4	5
c	10	12	14	16	18

Examining the table, we notice that c increases by 2 for every 1 that p increases. So $c = 2p +$ some number. The number can be found by tracing the table back to the point where p is 0. Since c decreases by 2 as we go toward the left, the term for $p = 0$ would be $c = 8$. Or we can obtain the number to be added by subtracting twice the value of p from the corresponding value of c, for if $c = 2p +$ a number, the number is $c - 2p$. Then our formula is $c = 2p + 8$.

If either of the letters is squared, the solution is much more difficult.

Class Exercises

In each of these exercises, the values of the second letter depend on those of the first letter. Make a simple formula from which the table could have been obtained.

1.

p	1	2	3	4	5
c	3	6	9	12	15

2.

b	0	1	2	3	4
A	0	2	4	6	8

3.

t	1	2	3	4	5
d	5	10	15	20	25

4.

x	0	2	4	6	8
y	0	4	8	12	16

5.

x	1	3	5	7	9
y	6	18	30	42	54

6.

p	1	2	3	4	5
c	8	9	10	11	12

7.

n	1	2	3	4	5
p	4	5	6	7	8

8.

x	0	2	4	6	8
y	3	5	7	9	11

9.

x	0	1	2	3	4
y	1	3	5	7	9

10.

x	0	1	2	3	4
y	5	8	11	14	17

11.

t	0	1	2	3	4
s	2	4	6	8	10

12.

w	1	2	3	4	5
f	7	10	13	16	19

13.

x	1	2	3	4	5
y	1	4	7	10	13

14.

x	1	2	3	4	5
y	− 1	1	3	5	7

Optional Exercises

15.

x	0	1	2	3	4
y	11	9	7	5	3

16.

x	0	1	2	3	4
y	5	0	− 5	− 10	− 15

17.

x	2	4	6	8	10
y	3	6	9	12	15

18.

t	0	3	6	9	12
r	0	5	10	15	20

19.

m	0	2	4	6	8
s	4	7	10	13	16

20.

x	0	5	10	15
y	− 5	− 1	3	7

21.

x	3	6	9	12
y	3	5	7	9

22.

x	2	4	6	8
y	− 3	− 6	− 9	− 12

23.

x	2	4	6	8
y	4	3	2	1

24.

x	3	6	9	12
y	11	7	3	− 1

Honor Work

25.

e	0	1	2	3	4
A	0	1	4	9	16

26.

e	0	1	2	3	4
V	0	1	8	27	64

27.

t	0	1	2	3	4
d	7	8	11	16	23

28.

t	0	1	2	3	4
s	0	5	20	45	80

29.

x	0	1	2	3	4
y	3	5	11	21	35

30.

x	0	1	2	3	4
y	− 1	0	3	8	15

31.

P	1	2	3	4	5
V	12	6	4	3	$2\frac{2}{5}$

32.

w	2	4	6	8	10
l	30	15	10	$7\frac{1}{2}$	6

33. In exercise 31, if you select any 2 pairs of corresponding values on P and V, is it then possible to rearrange them so as to form a proportion?

INVERSE VARIATION

The Lever

You have already used the lever in some of its forms. The teeter board or seesaw, the scissors, the scales for weighing, the crowbar for lifting weights, and the jack for raising your automobile—all are familiar forms of lever.

Here Rosemary and Margaret are riding on a teeter board. The board is resting on a support called the fulcrum. If Rosemary is heavier than Margaret, should she sit farther from the fulcrum or nearer to it? How should their distances from the fulcrum compare if Rosemary weighs just twice as much as Margaret? If just 3 times as much?

You can see that the heavier girl should sit nearer to the fulcrum. Perhaps you have also discovered that the product of the weight and distance of one girl equals the product of the weight and distance of the other.

Philip D. Gendreau, N.Y.

THE TEETER BOARD OR SEESAW

Formula: $w_1d_1 = w_2d_2$

Dividing both sides by w_2d_1:

$$\frac{w_1}{w_2} = \frac{d_2}{d_1}$$

Notice that in this proportion, the ratio of the d's has the sub-two on top whereas that of the w's has the sub-one on top. These quantities are then inversely proportional.

Two variables are said to ***vary inversely*** when their product is constant. Inverse variation may be expressed by the equation $y = \frac{k}{x}$ or $xy = k$.

Thought Questions

If x and y vary inversely, how does y change when x grows larger? If y decreases when x increases, do x and y necessarily vary inversely?

Exercises

1. Replace the blank with the word *directly* or *inversely:*

(*a*) The amount of interest I shall receive on an investment varies . . . as the time.

(*b*) The number of cubic feet of air per pupil in a classroom varies . . . as the number of pupils in the room.

(*c*) The number of pencils Dorothy can buy for $1 varies . . . as the price of a pencil.

(*d*) The distance Henry can run varies . . . as the time.

(*e*) The time it will take a car to run 1 mi. varies . . . as its speed.

(*f*) The income from a piece of property is $500 a yr. The per cent it will pay on the cost varies . . . as the cost.

(*g*) The cost of a number of pencils varies . . . as the price of each pencil.

2. Assuming that the quantities not mentioned are constant, state whether the variation is *direct* or *inverse:*

(*a*) The distance a car can go varies . . . as the rate.

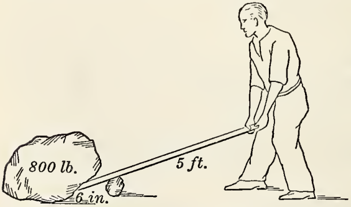
(*b*) The time the car takes varies . . . as the distance.

(*c*) The rate varies . . . as the distance.

(*d*) The rate varies . . . as the time.

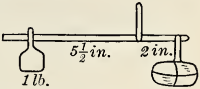
3. Katherine who weighs 60 lbs. is 4 ft. from the fulcrum of a teeter board. If Gladys, sitting at a distance of 5 ft. from the fulcrum, just balances her, what does Gladys weigh?

4. Emily weighs 80 lbs., and Grace weighs 70 lbs. If Grace sits 6 ft. from the fulcrum of a teeter board, how far from it must Emily sit to balance her?


800 lb.
5 ft.
6 in.

5. It requires 800 lbs. force to raise a rock. William sets his bar so that the fulcrum is 6 in. from the rock and 5 ft. from his hands. With what force must he press to raise the rock?

6. Sarah wishes to find out if the butcher gave her the right weight of meat. She has a 1 lb. weight, but no scales, so she suspends a light rod and hangs the meat 2 in. from the fulcrum. She finds that the 1 lb. weight, when suspended $5\frac{1}{2}$ in. from the fulcrum, balances the meat. What was the weight of the meat?



7. John wishes to exert a force of 600 lbs. on a rock with a bar 4 ft. long. If he is able to press down with a force of 120 lbs., how far from the rock should he place the fulcrum?

8. A teeter board is 12 ft. long. Two girls, weighing 70 lbs. and 50 lbs., want to sit on the ends of it. How far from the heavier should the support be placed?

9. Sarah is cutting tough cloth with a pair of scissors. Will it cut more easily when near the end of the scissors or near the fulcrum? Explain.

Inverse Variation in Science

Mr. Boyle compressed the gas in a tank and made this table showing how the volume changed as the pressure increased:

P	1	2	3	4	5	6
V	60	30	20	15	12	10

Before looking at Boyle's formula, see if you can write it yourself. As P increases, how does V change? Are the values of P and V proportional? Inversely proportional? Can V ever become 0?

The relation between the pressure and volume of a gas is given by the formula:

$$PV = K$$

This formula shows another way of expressing inverse variation.

1. Solve this formula for P, for V.
2. If $V = 10$ and $P = 8$, what is the value of K?
3. If $P = 6$ and $K = 54$, what is the value of V?

4. If $V = 16$ when $P = 9$, what is the value of V when $P = 24$? First find the value of K.

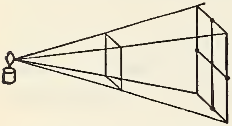
5. Find the value of V when:

(*a*) $P = 15$, if $V = 30$ when $P = 9$
(*b*) $P = 6$, if $V = 8$ when $P = 30$
(*c*) $P = 11$, if $V = 9$ when $P = 8$
(*d*) $P = 6.4$, if $V = 3.6$ when $P = 4.8$

6. In the formula $PV = K$, by what number is V multiplied when P becomes 5 times as great?

The Law of Inverse Squares

1. Since light from a point travels in straight lines, the same amount of light that would strike a figure 1 ft. away will strike at a distance of 2 ft. a similar figure whose corresponding sides are twice as long. How will their areas compare?



2. If the same amount of light is spread out over twice as great an area, we say that the intensity is half as great. How would the intensities compare in Exercise 1?

3. When you study your lesson in the evening, if you hold your book only 3 ft. from the light, how many times as great is the intensity of the illumination as it would be if you held the book 12 ft. from the light?

4. Write a formula for intensity of illumination using I_1 and I_2 for intensities and d_1 and d_2 for distances from the light.

5. As you move your book away from the light, does the intensity of illumination on it increase or decrease?

THE STRAIGHT LINE (Optional)

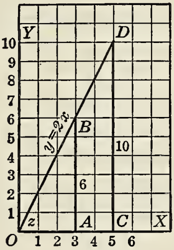
The line $y = mx$

Let us make a graph of the equation $y = 2x$. First we shall make the table:

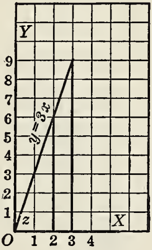
x	0	1	2	3	4	5
y	0	2	4	6	8	10

Are the corresponding values of x and y proportional? What is the ratio $y:x$?

What is the value of y when $x = 3$? When $x = 5$? Are the triangles OAB and OCD similar?



$y = 2x$



$y = 3x$

What is the ratio $\frac{AB}{OA}$? The ratio $\frac{CD}{OC}$? Are they equal?

Draw any other vertical line from OC to OD, and determine if the ratio for your line is the same as those you have already found.

Now make a graph of the equation $y = 3x$. Are the values of y and x proportional for this graph also? When you draw vertical lines as above, are the triangles similar? What is the ratio $\frac{y}{x}$ for this graph? Is the ratio the same for this graph $y = 3x$ as it was for $y = 2x$?

Plot the graph of $y = 4x$, of $y = x$, of $y = \frac{1}{2}x$. Then determine the ratio $\frac{y}{x}$ for each of these graphs.

Can you find any relation between the coefficient of x and the value of the ratio $y:x$ for each of these graphs?

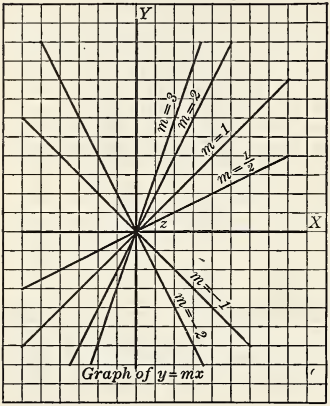
Slope of a line. When an equation is in the form $y = mx$, m, the coefficient of x, is the ratio of the y value to the

x value for any point of the line. It is called the slope of the line. Can you see why?

The slope of $y = mx$ is m

How does the line move as m increases?

What is the slope of $y = -2x$? Draw this line. What do you observe about its location?



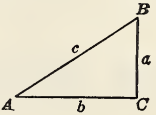
If z is the angle made by the line $y = mx$ with the x axis, how does z change when m grows larger? Can m change without z changing? Can z change without m changing?

If m decreases toward 0, how does z change? When $m = 0$, what is the value of z? When m becomes negative, what can you say about z?

Tangent of an angle. You have noticed that for each value of $\angle z$, the ratio $\frac{y}{x}$ or m has a definite value and does not change if you take large or small triangles, but that if $\angle z$ grows larger, this ratio grows larger too. This ratio $\frac{y}{x}$ or m is called the **tangent of the angle** (written $\tan z$).

In a right triangle, the ratio of the leg opposite an acute angle to the leg adjacent to the angle is the tangent of that acute angle.

$$\text{Tan } A = \frac{BC}{AC}$$



This is an important ratio about which you will learn more in the next chapter.

The line $y = mx + b$

We have just drawn the graph of $y = 2x$. Now suppose a number, say 3, were added, and our equation became $y = 2x + 3$, what change would it make in the graph? Making a table:

x	0	1	2	3	4	5
y	3	5	7	9	11	13

What is the slope of this line? Does adding a number change the slope of a line?

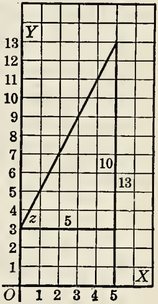
Plot the graphs of $y = 2x + 4$, of $y = 2x + 5$, of $y = 2x + 10$.

What effect does increasing the number have on the position of the graph? Does the line turn, or does it always make the same angle with the x axis? Does it move up or down? Does it remain parallel to its original position?

At what point is the y axis cut by the graph of $y = 2x$? Of $y = 2x + 3$? Of $y = 2x + 5$? Of $y = 2x + b$?

What is the relation between the value of b and the point where the graph cuts the y axis?

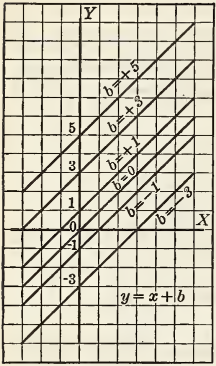
In the equation $y = mx + b$, m is the slope of the line or the tangent of the angle that the line makes with the x axis, and b is the point where the line cuts the y axis (the y intercept).



Thought Questions

If $AC = 10$ and $\tan A$ or $\dfrac{BC}{AC} = 2$, what is the length of BC? Do you think this a useful method of finding the length of a line?

How the line $y = mx + b$ moves when b grows larger. If we make graphs on the same axes of $y = x + b$ for different values of b, we get these lines. Are they parallel?



Can lines having the same slope meet? Let us see:

Solving $y = mx + 2$ and $y = mx - 1$ algebraically for their point of intersection we have:

$$\begin{array}{rl} y = & mx + 2 \\ \underline{y =} & \underline{mx - 1} \\ 0 = & \quad + 3 \end{array}$$

As it is impossible that 0 should equal 3, these equations cannot have a point in common. Consequently, lines having the same slope never meet, that is, they are parallel.

In the equation $y = mx + b$, if m does not change, then as b increases, we obtain a set of parallel lines.

Inconsistent equations. We have learned that equations are called inconsistent when they have no common solution. Any pair of parallel lines, therefore, have ***inconsistent equations.***

Exercises

Find the slope and the y intercept for each of these lines:

1. $y = 3x + 5$
2. $y = x + 1$
3. $y = 2x - 4$
4. $y = 4x - 2$
5. $y = -2x + 7$
6. $2y = 4x + 9$
7. $3y = x + 3$
8. $5y = -x + 7$
9. $2y = -5x - 8$
10. $3y = -2x - 4$
11. $2x - 3y = 10$
12. $3x + y = 7$
13. $5x - 3y = 12$
14. $x + y = 1$
15. $3x - 2y = -5$

Write the equation of the line if:

16. $m = 2$ and $b = 6$
17. $m = -3$ and $b = 1$
18. $m = -1$ and $b = -5$
19. $m = 0$ and $b = -5$

Tell which of the following lines are parallel:

20. $y = 2x + 5$ **23.** $y = 3x - 6$ **26.** $y = 3x$
21. $y = -x + 2$ **24.** $x + y = 7$ **27.** $2x - y = 3$
22. $y = -x - 4$ **25.** $2x - y = 9$ **28.** $x + y = -3$

Draw these lines by plotting the y intercept and using the slope:

29. $y = 3x - 5$ **31.** $y - x = 4$ **33.** $x + y = 8$
30. $y = x + 3$ **32.** $y + 2x = -2$ **34.** $x - 3y = 9$

Show how the point (x, y) must move if it always satisfies the equation:

35. $x + y = 4$ **37.** $x + 2y = 10$ **39.** $2x + 3y = 12$
36. $y = x$ **38.** $x - y = 5$ **40.** $y = 4$

Write the equation of a line parallel to the line whose equation is $y = 2x + 1$:

41. And passing through the origin.
42. And having its y intercept equal to 5.

Review Exercises

1. Separate 198 into two parts in the ratio 7:11.

2. Separate 153 into three parts in the ratio 3:5:9.

3. Solve for x:

(*a*) $\frac{x}{2} = \frac{3}{5}$ (*b*) $\frac{7}{x} = \frac{3}{8}$ (*c*) $\frac{x-1}{x+3} = \frac{9}{12}$ (*d*) $\frac{2x-5}{x+2} = \frac{2x-8}{x-1}$

4. Divide an estate of $46,000 among 4 heirs in the ratio 3:4:6:7.

5. Show whether these proportions are true:

(*a*) $\frac{3\frac{1}{7}}{4\frac{5}{7}} = \frac{14}{21}$ (*b*) $\frac{2.4}{4.2} = \frac{.8}{1.4}$

6. If concrete contains 1 part of cement and 3 parts of sand to 5 parts of gravel, how many pounds of cement will be required to make 450 lbs. of the mixture?

7. Write these equations as proportions:

(*a*) $ab = cd$ (*c*) $P_1V_1 = P_2V_2$
(*b*) $w_1d_1 = w_2d_2$ (*d*) $x^2 = ab$

8. Complete these tables, d being directly proportional to r:

(*a*)

r	2	5	12
d	14		

(*b*)

r	3	6	10
d	11		

9. Complete these tables, r being inversely proportional to t:

(*a*)

t	6	10
r	15	

(*b*)

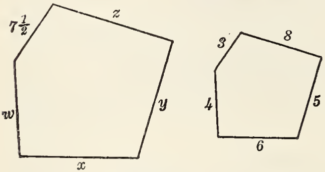
t	4	6	9
r	36		

10. A cylindrical pail 10 in. high holds 2 gals. when full. How many quarts of oil does it contain when it is filled to a depth of 4 in.?

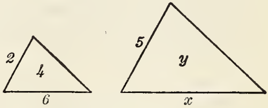
11. Are the values of y proportional to x in:

(*a*) $y = 3x$? (*b*) $y = 2x + 5$? (*c*) $x + y = 10$?

12. These polygons are similar. Find w, x, y, and z.



Ex. 12



Ex. 13

13. These triangles are similar.

(*a*) Find side x.

(*b*) If the area of the small triangle is 4, find the area of the large triangle.

14. Three men entered into partnership investing $8000, $10,000, and $12,000. How should they divide a profit of $4740?

15. A recipe calls for 3 c. of flour, 1 c. of sugar, and ½ c. of milk. How many pounds of sugar are used in making 3 lbs. of the product?

16. Potassium nitrate (K + N + O 3) is used in explosives because of the large amount of oxygen it contains. If K = 39, N = 14, and O = 16, how much oxygen is there in 1010 lbs. of potassium nitrate?

17. A picture is in the form of a root 5 rectangle. If the length must be reduced to 4 in. to fit a page, to what will the width be reduced?

18. Which of these musical intervals are harmonious? C = 256, D = 288, E = 320, G = 384, and B = 480.

(*a*) C and D? (*c*) D and E? (*e*) D and B?
(*b*) C and E? (*d*) D and G? (*f*) B and C?

19. What is the modified scale, and why was it needed on the piano?

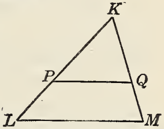
20. When you blow into a harmonica, the notes produced are C, E, G, and C. Can you produce a discord by any combination of these notes?

21. On a certain map a distance of 300 mi. is represented by a length of 5 in. What distance does 3 in. represent? How many inches represent a distance of 240 mi?

22. A water pipe has a circumference of 3 in. What is the circumference of another water pipe whose diameter is twice as large as that of the first pipe?

23. PQ is parallel to LM. Find the length of QM:

(*a*) If $KP = 4$, $PL = 5$, and $KQ = 6$
(*b*) If $KP = 7$, $PL = 3$, and $KM = 20$
(*c*) If $KL = 12$, $PL = 5$, and $KM = 9$
(*d*) If $KP = a$, $PL = b$, and $KQ = e$



24. A side of a triangle is 10. Find the corresponding side of a similar triangle whose area is 9 times as large.

25. Two angles of one triangle are 50° and 55°. Two angles of another triangle are 50° and 75°. Are these triangles similar? How many degrees are there in the other angles?

26. How high is a building that casts a shadow 42 ft. long when a yardstick casts a shadow 42 in. long?

27. If your camera is 5 in. deep, how far from a person 6 ft. tall should you stand to take a picture of him 4 in. high?

28. Tell whether these quantities vary directly or inversely:

(*a*) The rate a person walks and the distance he goes in a certain time (*b*) The rate a man walks and the time it takes to go a certain distance

(*c*) The interest rate on an investment and the income from it

(*d*) The interest rate on an investment and the principal that must be invested to produce a certain income

29. Make formulas showing the relations in these tables:

(*a*)

x	1	2	3	4
y	5	10	15	20

(*c*)

n	1	2	3	4
s	5	8	11	14

(*b*)

h	1	2	3	4
k	5	6	7	8

(*d*)

r	1	2	3	4
d	1	6	11	16

30. Robert wants to raise a stone with an iron bar. He places the support 4 in. from the stone and pushes down on the bar with a force of 90 lbs. at a distance of 5 ft. from the support. What force is he exerting on the stone?

31. A straight line passes through the point (0, 0).

(*a*) If $m = 3$, what is the equation of the line?

(*b*) If it also passes through (3, 3), what is its slope?

(*c*) If it passes through (2, 5), what value has y when $x = 4$?

32. Find the slope and the y intercept of these lines:

(*a*) $y = 5x + 3$ (*b*) $y = 2x - 7$ (*c*) $y = x$

33. A moving picture film has pictures $\frac{3}{4}$ in. by 1 in. If this is thrown on a screen 9 ft. by 12 ft., what size will a man, $\frac{1}{2}$ in. high in the film, appear on the screen?

34. Mr. Grant earns \$42 a week and saves \$8 of it. Find the ratio of his savings to his earnings. If Mr. Cushman, who earns \$28 a week, wants to save in the same ratio, how much a week must he save?

35. Using the formula $m = \dfrac{c}{(1 - p)(1 - d)}$, find the price at which a dealer must mark a gas stove that cost \$31.50 so he can give a discount of 40% and still make a profit of 30% of the selling price. Check by subtracting the discount and profit from the marked price.

Test on Ratio

1. Divide 48 in the ratio 9:7.
2. Divide 100 into three parts in the ratio 5:3:2.
3. Write $km = np$ as a proportion.

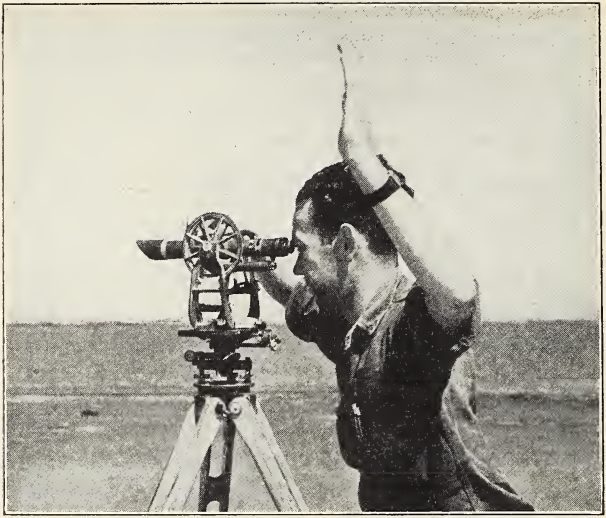
Solve, using the methods of proportion:

4. $\frac{5}{X} = \frac{4}{3}$ **5.** $\frac{x}{x - 3} = \frac{5}{4}$ **6.** $\frac{x - 5}{x + 3} = \frac{1}{2}$

7. If 100 ft. of wire weighs 8 lbs., what is the weight of 350 ft.?

Are the values of y proportional to those of x in:

8. $y = 2x + 5$? **9.** $y = 3x$? **10.** $y = x$?



By Ewing Galloway, N.Y.

THE TRANSIT

Chapter 13

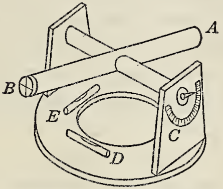
INDIRECT MEASUREMENT

SURVEYING INSTRUMENTS

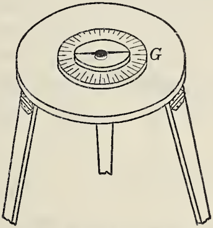
THE instrument shown in the picture is a ***transit.*** It is used by surveyors and engineers for laying out streets, railroads, boundaries of fields and plots for buildings, and for other work in which the accurate measurement of angles is needed. To measure angles in a vertical plane, the ***telescope*** is tipped up or down, and the number of degrees is read on the large circle that you see on this side of the instrument. The long tube just under the telescope is a ***level.*** It is used to determine the horizontal line from which the angle is to be measured. The telescope can also be turned to the right or left to measure angles in a horizontal plane. The number of degrees is read on a large circle some distance below the telescope. In the center of this circle is a *compass* by means of which the surveyor can determine the number of degrees from a north and south line.

The transit is an expensive instrument, for the surveyor must make his measurements very accurately. In the measurements that you will want to make, however, such as the height of a pole or the distance across a river, an error of a few inches or even feet will not matter. And for this purpose you can make a cheap instrument yourself.

How to make a transit. First, make a frame like that shown here. If you make it of a single piece of metal with the ends bent up to form the supports for the telescope, it will be stronger and more usable than if you make it of wood. It should have a circular hole in the center for the scale on the frame to come through. Next, fasten a protractor to the frame, as shown at *C*, and two medicine vials nearly filled with water, as at *D* and *E*. Next, take a cheap telescope or just a tube, and put cross wires on one end, as at *B*. Attach this strongly to an axis that should be pivoted to the frame with a pointer on the end to move over scale *C*. This pointer must be parallel to the tube, so it will show the angle made by the tube with the horizontal.



THE TRANSIT FRAME



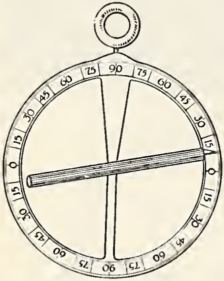
THE TRANSIT TABLE

The second part of the instrument, on which the part already made rests, consists of a horizontal circle supported by three legs hinged to its under side. In the center, put two protractors together to form a circle, as at *G*, and in the center of this circle a small compass.

Another design, that is easy to make, places the levels on the second part of the instrument. The telescope is supported in a fork whose shaft passes down through the center of the second part. The pointer is then attached to the shaft of the fork.

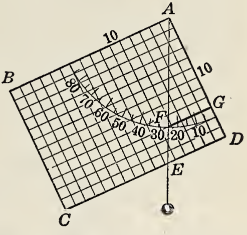
If you are a little ingenious, perhaps you can make a better transit of your own design.

How to make an astrolabe. A much simpler but very useful instrument for measuring angles in a vertical plane is the ***astrolabe.*** This instrument was much used by early explorers, and one lost by Samuel Champlain 300 yrs. ago was recently found. If you can get the face of an old clock, you will have ready-made a scale laid off in arcs of 6°. Attach a ring to one side of it, and if it is too light to hang down satisfactorily, attach a weight to the opposite side. Now fasten a small tube so that it will turn around the point through which the axis of the hands passed, and make a pointer at one end of the tube to pass over the scale. Mark 90° directly under the ring so that the line from 0° to 180° will be horizontal.



To measure an angle, hold the astrolabe by the ring, and sight along the tube. The number of degrees is then read where the pointer crosses the scale. The astrolabe can also be used for measuring horizontal angles by being placed on a horizontal table such as the lower part of the transit.

How to make a clinometer. Take a sheet of graph paper, and with one corner as a center and a radius of 10 squares, draw a quarter circle. Lay off the arc in degrees as shown in the figure. Paste this sheet on a board or heavy cardboard, and suspend a weight from the center of the circle.

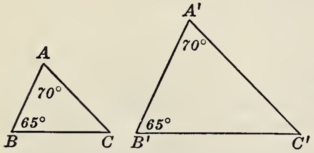


When you have sighted along the *clinometer,* place a finger on the cord before lowering the instrument. This retains the angle for you so you can read it after lowering the clinometer.

The sextant. This instrument is used in navigation both on the sea and in the air. The observer looks at the horizon through his telescope and at the same time sees an image of the sun or of a star reflected by mirrors. One mirror is attached to a movable part. As this part, with its mirror, is turned, a pointer moves over a scale. When the sextant is adjusted so that the image seems to touch the horizon, the angle of the sun or star above the horizon is shown on the scale. The observer uses this to help locate his position on the earth.

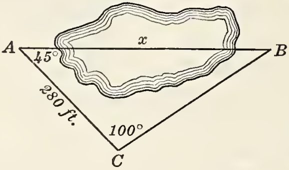
SCALE DRAWING

Surveyors and engineers often find a distance by drawing a figure similar to that to be measured and then finding the distance by proportion. You remember that in similar figures, corresponding lines are proportional.



How to make triangles similar. We can make a triangle similar to another triangle if we make two of its angles equal to those of the other triangle. Triangles *ABC* and $A'B'C'$ are similar because $\angle A$ and $\angle A'$ are each 70° and $\angle B$ and $\angle B'$ are each 65°.

Right triangles are similar if we make one acute angle of one of them equal to an acute angle of the other, for since the right angles are equal, they have then two angles of one equal to two angles of the other. In finding heights the right triangle is very useful.



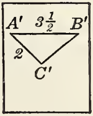


Illustration. To find the distance from A to B across a pond, the Boy Scouts set up a transit at A and sighted across to B. Then they turned the telescope and sighted C. They found from the scale that $\angle A$ was 45°. In the same way they found that $\angle C$ equaled 100°. Next they measured the length AC. This they found to be 280 ft. Next they drew on paper a triangle in which $\angle A' = 45°$, $A'C' = 2$ in., and $\angle C' = 100°$. They found that $A'B'$ was $3\frac{1}{2}$ in. long. Finally, they wrote a proportion and solved it for x. Here is their computation:

$$\frac{x}{3\frac{1}{2}} = \frac{280}{2}$$

Multiplying by $3\frac{1}{2}$: $$x = \frac{3\frac{1}{2} \times 280}{2}$$

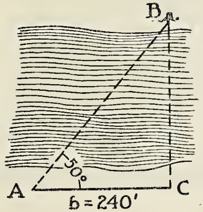
$$= 490 \text{ or about } 500 \text{ ft.}$$

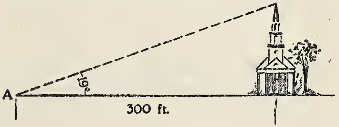
HISTORICAL NOTE ON TRIGONOMETRY: Practical trigonometry goes back to Hipparchus (140 B.C.), a Greek astronomer, who worked out trigonometric tables for use in astronomy. In 1464 Regiomontanus established trigonometry as an independent science. During the past 300 years, the development of algebraic symbols has aided the analytic treatment of trigonometric functions.

Scale Drawing for the Boy Scout

To solve these problems, draw on paper with protractor and ruler a triangle similar to the one given in the exercise. Then measure the sides of your triangle, and make a proportion using your measurements and the distance given.

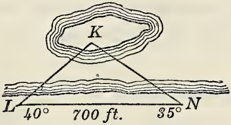
1. To find the distance across a river from C to B, two Boy Scouts made a right angle at C, measured AC 240 ft., and found $\angle A$ was then 50°. Find the distance across the river. As a convenient scale, use 1 in. = 80 ft.





2. To find the height of her church, Emily measured $\angle A$ and the distance from A to the church. If the distance was 300 ft. and $\angle A$ was 19°, what was the height of the church? As a scale, use 1 in. = 100 ft.

3. Two Scouts wanted to know the distance across a river, so they measured 2 angles and a side. They found $\angle A = 60°$, $\angle C = 90°$, and $AC = 300$ ft. From a scale drawing find the distance across the river. (The figure is the same as that of Ex. 1.)

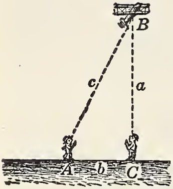


Ex. 4

4. To measure the distance from a point L on the shore to an island K, Richard found that $\angle L = 40°$, $\angle N = 35°$, and $LN = 700$ ft. Make a scale drawing using 1 in. = 200 ft., and from it find the distance KL.

5. Chester who is at C observes that an airplane is directly

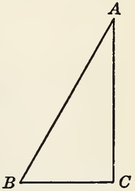
overhead at the same time that Tom 800 yds. away at A finds the angle of elevation ($\angle A$) to be 62°. Choose a suitable scale, and find the height of the airplane from a scale drawing.



6. To find the distance from A to B across a pond, Paul measured AC 300 yds. directly south. Then he found that B was directly west 180 yds. What was the distance from A to B?

7. Find the height of a building AC if $BC = 20$ ft. and $\angle B = 70°$.

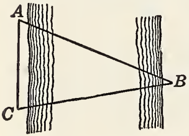
8. Find the height of a pole AC if BC is 12 ft. and $\angle B$ is 58°.



9. Find the height of a tree AC if its shadow BC is 30 ft. and $\angle B$ is 48°.

Find the distance across the river from A to B if:

10. $\angle A = 60°$, $\angle C = 70°$, and $AC = 200$ ft.



11. $\angle A = 80°$, $\angle C = 75°$, and $AC = 450$ ft.

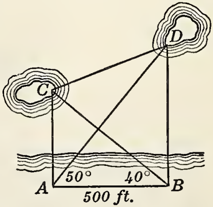
12. $\angle A = 70°$, $\angle C = 60°$, and $AC = 300$ ft.

13. A baseball diamond is a square 90 ft. on a side. Make a scale drawing, and find the distance from third base to first.

14. Make a scale drawing of the lot on which your home stands, putting in the outline of the house.

15. Make a floor plan of your house or apartment.

16. Robert told Richard that he could find the distance from island C to island D without leaving the shore. To prove it, he measured a line AB on the shore 500 ft. long at right angles to the lines AC and BD. Then he found that the $\angle CBA$ was 40° and $\angle DAB$ was 50°. What distance did he find CD to be?



TRIGONOMETRY: USE OF THE TRIANGLE IN MEASUREMENT

Mt. Everest in Asia is 29,002 ft. high. The moon is 238,000 mi. away. The area of the United States is 3,026,789 sq. mi. These are all well-known facts, but how were they determined? No one has yet climbed to the top of Mt. Everest. Certainly no one has been to the moon. And the 3,000,000 sq. mi. in the United States were not determined by stretching a tape measure around the country side. Evidently some other way of doing this was found. We shall now learn how such measurements are made.

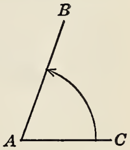
Have you ever watched a surveyor at work? He uses a transit * for measuring angles of triangles, and then from these triangles he determines the distances. He could, and sometimes does, make a scale drawing, but for much of his work this method is not accurate enough. So he uses the method you are to learn now.

Because they have no stationary place for setting up a transit, the ship captain on the sea and the aviator in the air determine their location by using another instrument, the sextant. To understand how they work, we must understand triangles.

Some facts we already know about angles and triangles.

1. An angle is formed when two lines meet. We generally read it by a capital letter placed near its vertex.

2. We may think of $\angle A$ as having been formed by turning the line AB around point A from the position AC to the position AB. The size of the angle depends on the amount of this turning and not on the length of the sides.



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* An inexpensive transit and sextant, accurate enough for your measurements, is made by the Educator Toy Co., Bronxville, N.Y.

3. The number of degrees in an angle can be read by placing the vertex at the center of a circle that is divided into degrees and reading the numbers where the sides cross the circle.

4. The sum of the angles of a triangle is a straight angle or 180°.

5. A right angle equals 90°.

6. The sum of the two acute angles of a right triangle is 90°.

7. Vertical angles are equal.

8. Alternate interior or corresponding angles of parallel lines are equal.

9. Two triangles are similar if two angles of one of them equal two angles of the other.

10. Two right triangles are similar if an acute angle of one equals an acute angle of the other.

11. The corresponding sides of similar triangles are proportional.

Reviewing the Triangle

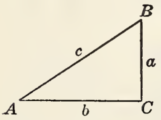
1. If the three angles of a triangle are all equal to each other, how many degrees are there in each of them?

In triangle ABC, ∠C is the right angle.

2. How many degrees are there in $\angle B$ if $\angle A$ equals: 30°? 45°? 25°? 38°? 57°? 89°?

3. If $\angle A$ grows larger, how does $\angle B$ change?

4. Find the number of degrees in $\angle A$ if:

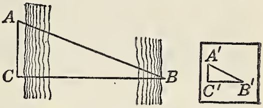


(*a*) $\angle A = \angle B$
(*b*) $\angle A$ is 20° larger than $\angle B$
(*c*) $\angle A$ is 4 times as large as $\angle B$
(*d*) $\angle A$ is 10° more than twice $\angle B$
(*e*) $\angle A$ is to $\angle B$ as 7 to 11

5. Find the number of degrees in each angle of a triangle if they are in the ratio 2:3:4. In the ratio 11:16:9.

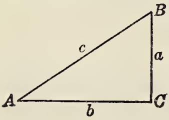
6. Find the number of degrees in each angle of the triangle ABC if $\angle B$ is 30° more than $\angle A$, and $\angle C$ is 10° more than twice $\angle B$.

A new way to use ratios to find distances. Paul wished to find the distance across the Otter Creek from C to B, so he measured a distance of 100 ft. in the direction CA at right angles to CB, and then he found with a protractor that $\angle A$ was 78°. He next drew carefully on paper a right $\triangle A'B'C'$, having $\angle A'$ equal to 78°, and measured $C'B'$ and $A'C'$. He found that, if he made $A'C'$ 2 in. long, $C'B'$ was about 9.4 in. long. Paul concludes that, since $C'B'$ is 4.7 times as long as $A'C'$, CB must be 4.7 times as long as AC or 470 ft., because $\triangle$s ABC and $A'B'C'$ are similar.



Now Paul measured only one line on the ground and two short lines on paper. And he measured the lines $C'B'$ and $A'C'$ on paper only to determine the ratio of CB to AC. Evidently, if he had known this ratio beforehand, he would have needed to measure only one line. Since all right triangles, having an $\angle A$ equal to 78°, are similar, this ratio could be computed once for all for a 78° angle. Therefore it would not be necessary to draw a triangle on paper every time we wished to find the ratio. Then, however large or small $\triangle ABC$ is, this ratio will remain unchanged so long as the size of $\angle A$ remains unchanged. But if $\angle A$ grows larger or smaller, the ratio will change with it. This ratio is, therefore, called a *function* of $\angle A$.

In $\triangle ABC$, in which $\angle C$ is a right angle, the ratio of the leg opposite an acute angle to the leg adjacent to that angle is the ***tangent of the angle***, written *tan A* or *tan B*.



$$\text{Tan } A = \frac{a}{b} = \frac{\text{opposite leg}}{\text{adjacent leg}}$$

$$\text{Tan } B = \frac{b}{a} = \frac{\text{opposite leg}}{\text{adjacent leg}}$$

The tangent of B is called the cotangent of A, written *cot A*.

Exercises

1. Using a protractor, draw a right triangle having $\angle A$ equal to 10°. Measure a and b carefully and compute the value of tan 10° to the nearest hundredth.

2. From the same triangle compute tan 80°.

3. By drawing a triangle having $\angle A$ equal to 20°, compute tan 20° and tan 70°.

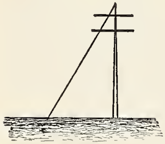
4. Using the same method, fill the blanks in the following table:

Angle	*Tan*	*Angle*	*Tan*
10°		50°	
20°		60°	
30°		70°	
40°		80°	

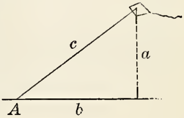
5. As the angle increases from 10° to 80°, does its tangent increase or decrease?

6. Find tan 45° without measuring lines.

7. Henry finds that a wire from the top of a telegraph pole touches the ground 20 ft. from the foot of the pole and makes an angle of 50° with the ground. Using your table, find the height of the pole.



If we know any side of a right triangle and an acute angle, we can find the other two sides by using the tangent ratio, but sometimes this method would be very inconvenient, as in the following example:



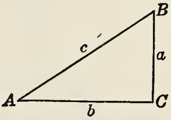
Robert is flying his kite. After letting out 100 ft. of string, he notices that the string makes an angle of 40° with the ground. Assuming that the string is straight, how high is the kite? Robert says that he could determine the height very easily if he knew the ratio of a to c.

Evidently to solve problems such as this, we need another ratio, that of the opposite side to the hypotenuse.

In a right triangle, the ratio of the side opposite an acute angle to the hypotenuse is the ***sine of the angle,*** written *sin A* or *sin B*.

$$\text{Sin } A = \frac{a}{c} = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\text{Sin } B = \frac{b}{c} = \frac{\text{opposite side}}{\text{hypotenuse}}$$



The sine of B is called the *cosine of A*, written *cos A*.

Notice that in both the sine and the tangent, the side opposite the angle is always the numerator. If the denominator is the hypotenuse, the ratio is the sine; if it is not the hypotenuse, the ratio is the tangent.

Exercises

1. Draw a right triangle having $\angle A$ equal to 10° and measure a and c carefully. From their lengths, find sin 10° to the nearest hundredth.

2. From the same triangle, find sin 80°.

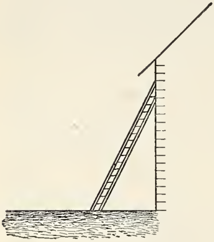
3. By drawing right triangles and measuring their sides as in the last group of exercises, fill the blanks in the following table:

Angle	*Sin*	*Angle*	*Sin*
10°		50°	
20°		60°	
30°		70°	
40°		80°	

4. Using the value from your table, find the height of Robert's kite.

5. Make a table for cosines like that for sines in Exercise 3. How can you find the cosine of an angle from a table giving only sines of angles?

6. John has a ladder 12 ft. long on which he wishes to climb to the eaves of his house. If for safety, the ladder must make an angle of 70° with the ground, how high will it reach? Use your table.



How to use the tables. You have already made a two-place table of the sines and tangents of angles at intervals of 10°. In the same way you could have filled in these functions for angles differing by one degree. On page 324 such a table is given. In the first column is the number of degrees in the angle. In the second column, opposite the number of degrees, is the sine of the angle, and in the fourth column the tangent of the angle is given.

Illustration 1. Find tan 18°. Look down the column headed degrees until you come to 18. The number .3249 found in the tangent column opposite 18 is the answer.

Illustration 2. Find sin 37°. The required answer is the number .6018 found in the sine column opposite 37.

Exercises

Use the table on page 324 for these exercises:

1. Verify the following:

(*a*) sin 13° = .2250
(*b*) tan 77° = 4.3315
(*c*) tan 24° = .4452
(*d*) sin 86° = .9976
(*e*) cos 65° = .4226
(*f*) cos 20° = .9397

2. Find the value of the following functions:

(*a*) tan 25°
(*b*) tan 43°
(*c*) sin 16°
(*d*) cos 37°
(*e*) sin 75°
(*f*) tan 5°
(*g*) sin 31°
(*h*) sin 45°
(*i*) tan 1°
(*j*) cos 51°
(*k*) sin 10°
(*l*) sin 68°
(*m*) tan 70°
(*n*) tan 89°
(*o*) cos 83°
(*p*) tan 3°
(*q*) cos 12°
(*r*) tan 41°
(*s*) sin 72°
(*t*) sin 56°

3. Verify these angles:

(*a*) $\sin x = .7660 \quad x = 50°$
(*b*) $\sin x = .9962 \quad x = 85°$
(*c*) $\tan x = 28.6363 \quad x = 88°$
(*d*) $\tan x = .5543 \quad x = 29°$

4. Find the value of x for these functions:

(*a*) $\sin x = .4540$
(*b*) $\tan x = .7002$
(*c*) $\tan x = 2.1445$
(*d*) $\sin x = \frac{1}{2}$
(*e*) $\sin x = .8090$
(*f*) $\tan x = 1$

5. For what values of x is $\sin x$ nearly equal to $\tan x$?

6. For what value of x do $\sin x$ and $\tan x$ differ most?

7. For what value of x is $\tan x$ about twice $\sin x$?

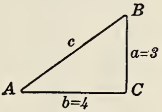
8. Is the sine of a 60° angle twice the sine of a 30° angle? Is tan 80° twice tan 40°?

9. In a right triangle, the side opposite an angle of 64° is . . . times the side adjacent to the angle.

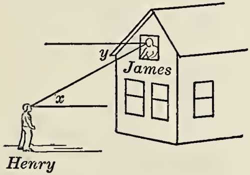
10. In a right triangle, the side opposite an angle of 22° is . . . times the hypotenuse.

11. In the right triangle ABC, a is 3 and b is 4.

(*a*) $\tan A$ is . . . .
(*b*) $\sin A$ is . . . .
(*c*) side c is . . . .
(*d*) $\tan B$ is . . . .
(*e*) $\sin B$ is . . . .
(*f*) $\cos A$ is . . . .



12. In the right triangle ABC, $\angle A = 45°$. Without the table find $\tan A$, $\tan B$, $\sin A$, $\sin B$.



Angle of elevation. Henry, who stands on the ground, is looking up at James, who is at an upstairs window. The angle x that the line of sight from Henry to James makes with the horizontal line is called the *angle of elevation.*

Angle of depression. The angle y that the line of sight from James to Henry makes with the horizontal is called the *angle of depression.*

TRIGONOMETRIC TABLE

Angle	*Sin*	*Cos*	*Tan*	*Angle*	*Sin*	*Cos*	*Tan*
1°	.0175	.9998	.0175	**46°**	.7193	.6947	1.0355
2°	.0349	.9994	.0349	**47°**	.7314	.6820	1.0724
3°	.0523	.9986	.0524	**48°**	.7431	.6691	1.1106
4°	.0698	.9976	.0699	**49°**	.7547	.6561	1.1504
5°	.0872	.9962	.0875	**50°**	.7660	.6428	1.1918
6°	.1045	.9945	.1051	**51°**	.7771	.6293	1.2349
7°	.1219	.9925	.1228	**52°**	.7880	.6157	1.2799
8°	.1392	.9903	.1405	**53°**	.7986	.6018	1.3270
9°	.1564	.9877	.1584	**54°**	.8090	.5878	1.3764
10°	.1736	.9848	.1763	**55°**	.8192	.5736	1.4281
11°	.1908	.9816	.1944	**56°**	.8290	.5592	1.4826
12°	.2079	.9781	.2126	**57°**	.8387	.5446	1.5399
13°	.2250	.9744	.2309	**58°**	.8480	.5299	1.6003
14°	.2419	.9703	.2493	**59°**	.8572	.5150	1.6643
15°	.2588	.9659	.2679	**60°**	.8660	.5000	1.7321
16°	.2756	.9613	.2867	**61°**	.8746	.4848	1.8040
17°	.2924	.9563	.3057	**62°**	.8829	.4695	1.8807
18°	.3090	.9511	.3249	**63°**	.8910	.4540	1.9626
19°	.3256	.9455	.3443	**64°**	.8988	.4384	2.0503
20°	.3420	.9397	.3640	**65°**	.9063	.4226	2.1445
21°	.3584	.9336	.3839	**66°**	.9135	.4067	2.2460
22°	.3746	.9272	.4040	**67°**	.9205	.3907	2.3559
23°	.3907	.9205	.4245	**68°**	.9272	.3746	2.4751
24°	.4067	.9135	.4452	**69°**	.9336	.3584	2.6051
25°	.4226	.9063	.4663	**70°**	.9397	.3420	2.7475
26°	.4384	.8988	.4877	**71°**	.9455	.3256	2.9042
27°	.4540	.8910	.5095	**72°**	.9511	.3090	3.0777
28°	.4695	.8829	.5317	**73°**	.9563	.2924	3.2709
29°	.4848	.8746	.5543	**74°**	.9613	.2756	3.4874
30°	.5000	.8660	.5774	**75°**	.9659	.2588	3.7321
31°	.5150	.8572	.6009	**76°**	.9703	.2419	4.0108
32°	.5299	.8480	.6249	**77°**	.9744	.2250	4.3315
33°	.5446	.8387	.6494	**78°**	.9781	.2079	4.7046
34°	.5592	.8290	.6745	**79°**	.9816	.1908	5.1446
35°	.5736	.8192	.7002	**80°**	.9848	.1736	5.6713
36°	.5878	.8090	.7265	**81°**	.9877	.1564	6.3138
37°	.6018	.7986	.7536	**82°**	.9903	.1392	7.1154
38°	.6157	.7880	.7813	**83°**	.9925	.1219	8.1443
39°	.6293	.7771	.8098	**84°**	.9945	.1045	9.5144
40°	.6428	.7660	.8391	**85°**	.9962	.0872	11.4301
41°	.6561	.7547	.8693	**86°**	.9976	.0698	14.3007
42°	.6691	.7431	.9004	**87°**	.9986	.0523	19.0811
43°	.6820	.7314	.9325	**88°**	.9994	.0349	28.6363
44°	.6947	.7193	.9657	**89°**	.9998	.0175	57.2900
45°	.7071	.7071	1.0000				

Notice that the angles of elevation and depression are the same size, for they are alternate interior angles of the two horizontal lines. The only difference is the position of the observer. If you are below looking up, it is an angle of elevation, but if you are above looking down, it is an angle of depression.

How to find distances by using the tangent and sine. If you know the length of one side of a triangle and the ratio of that side to another side, you can find the length of the other side. Therefore, to find a side, you must choose that function of the angle which is the ratio of the known side and the side to be found.

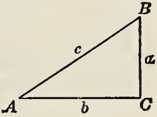
Illustration 1. If $\angle A = 40°$ and $AC = 200$ ft., find BC.

You must select that function of $\angle A$ that is the ratio of BC to AC. That function is tan A. So write:

$$\frac{x}{200} = \tan A$$

Since $\angle A$ is 40°, tan A is .8391, so we have

$$\frac{x}{200} = .8391$$



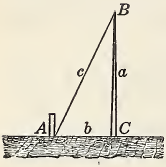
Now solve your equation for the unknown side, and perform the multiplication and division.

$$\begin{aligned} x &= 200 \times .8391 \\ &= 167.82 \\ &= 168 \text{ ft. to the nearest foot} \end{aligned}$$

To check, find $\angle B$. Then find side b by using the values of a and of tan B just found. Compare this result with the given value of b.

Illustration 2. Mr. White wishes to brace an antenna pole in his back yard by fastening a wire from its top to a post 23 ft. from the foot of the pole. He finds that the angle at A is 41°. What length of wire will he need?

Here b is 23 and $\angle A$ is 41°. To find c. The ratio $\frac{b}{c}$ is the sine of $\angle B$. So we must first find $\angle B$.



$$\begin{aligned} \angle B &= 90° - \angle A \\ &= 90° - 41° \\ &= 49° \\ \sin B &= \frac{b}{c} \end{aligned}$$

Multiplying by c: $c \sin B = b$

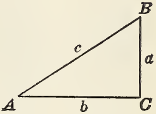
Dividing by $\sin B$: $c = \dfrac{b}{\sin B}$

Since b is 23 and sin 49° is .75 to two significant figures,

$$c = \frac{23}{.75} = 31$$

Exercises

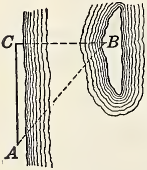
In triangle ABC, $\angle C$ is a right angle.



1. If $c = 30$ and $\angle A = 28°$, find b.
2. If $c = 10$ and $\angle B = 53°$, find b.
3. If $b = 2.5$ and $\angle A = 75°$, find a.
4. If $a = 20$ and $\angle A = 35°$, find b.
5. If $b = .42$ and $\angle B = 20°$, find c.
6. If $a = 350$ and $\angle B = 84°$, find c.

Trigonometry for the Scout

1. Girl Scouts, who planned to conduct a swimming contest from point C on the shore to point B on an island, found the distance as follows. They measured CA 340 ft. along the shore at right angles to CB, and then found that $\angle A$ was 38°. What is the distance from C to B?

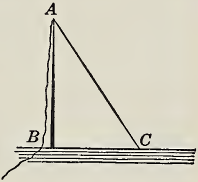


2. To find the distance CB across a river, the Boy Scouts measured a line AC at right angles to CB which they found to be 120 ft. Then with a transit they found $\angle A$ to be 82°. Find for them the length of CB.

3. A steep mountain road makes an angle of 36° with the horizontal. How many feet have I risen when I have traveled 385 ft. up the road?

4. The angle of elevation of the top of a mountain known to be 6000 ft. high is 19°. How far is the top of the mountain from the observer?

5. A rope 80 ft. long from the top of a flag-pole makes an angle of 75° with the ground when pulled tight. Find the height of the pole.



6. Sarah found the height of her home by noting that it cast a shadow 24 ft. long when the angle of elevation of the sun was 57°. How high was her home?

7. A tower on the bank of a river is known to extend 163 ft. above the water. If the angle of elevation of the top from a point on the opposite bank is 15°, how wide is the river?

8. If the angle of elevation of the top of a tower 800 ft. away is 9°, what is the height of the tower?

9. Charles' father told Charles that it was not safe to climb a ladder if it made an angle of more than 65° with the ground. How far from the house can Charles place the foot of a ladder 14 ft. long if it is to lean against the house?

10. The Boy Scouts climbed a mountain 4000 ft. high. From its top they saw a lake at an angle of depression of 18°. How far from them in a straight line is the lake?

11. Some boys cut 12-ft. poles to build themselves a wigwam. If the poles meet the ground at an angle of 68°, how high is the center point of the wigwam?

12. From the top of a cliff 300 ft. high, the angle of depression of a ship is 36°. How far is the ship from the foot of the cliff?

13. John found that from his classroom window the angle of depression of the opposite side of the street was 14°. If the window is 24 ft. above the ground, how far is it in a straight line from the window to the opposite side of the street?

14. At noon on June 21, a certain flag-pole 100 ft. high casts a shadow 30 ft. long. What is the angle of elevation of the sun?

Courtesy of Boy Scouts of America.

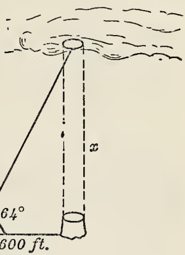
BOY SCOUTS MAKING A MAP

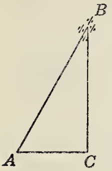
These boys are using a plane table and an alidade for plotting points on their map. The plane table is a drawing board set up on three legs. The alidade is a ruler with upright sights. The boy sights a point along the alidade and then draws the line along the edge on the paper. After sighting all the required points in this way and drawing lines, he moves his plane table to another position and sights the same points again. The intersections of the lines to the same point locate that point on his map.

Trigonometry for the Practical Man

1. A forest ranger has an observation post on top of a mountain 3816 ft. higher than the surrounding territory. He sees smoke rising from a fire in the forest at an angle of depression of 11°. How far away in a straight line is the fire?

2. The "airman's ceiling" is the under side of the clouds as seen from the earth. To find its height at night, the observer causes a powerful light to throw its rays vertically upwards. Then from his post 600 ft. from the light, he reads the angle of elevation of the spot of light thrown on the cloud. If he finds this angle to be 64°, what is the height of the "airman's ceiling"?

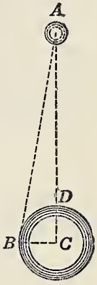




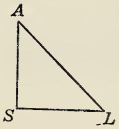
3. Later when the clouds lower, he finds that the angle of elevation has become only 18°. What now is the height of the airman's ceiling? What angle of elevation must the observer have found, if at one time he reported "ceiling 3000 ft."? "Ceiling 500 ft."? "Ceiling zero"?

4. Two artillery officers want to locate the position of the enemy's battery by the flash of the guns. Lieut. Homer observes that $\angle BCA$ is a right angle, whereas Capt. Meyer, 468 ft. from him at A, finds that $\angle BAC$ is 82°. How far is the battery B from C?

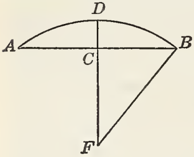
5. An astronomer at D observes that the moon is directly overhead. Six hours later, when the earth has turned so that he is at B, he finds $\angle CBA$ to be 89°. If the radius of the earth is about 4000 mi., how far is the moon from B?



6. The captain of a ship, S, observed that a lighthouse, L was directly east. After sailing 5 mi. north, he noticed that the lighthouse was 33° east of south ($\angle A = 33°$). How far was he from the lighthouse at first?

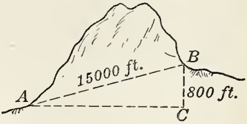


7. A railroad engineer needs to know the radius of a curve so he will be able to raise the outside rail the right amount to keep the train from jumping from the track. He cannot measure it directly as the track passes between a cliff and a river, so he determines that an arc of 20° has a chord AB 340 ft. long. Find FB the radius of the curve ($\angle F$ = 10° and $BC = \frac{1}{2} AB$).



8. A war observer in a captive balloon at a height of 2000 ft. above the earth finds that the enemy is concentrating at a point whose angle of depression is 22°. How far away from the war observer is the enemy?

9. Two towns A and B are on opposite sides of a mountain range and are 15,000 ft. apart in a straight line. The altitude of B above sea level is 800 ft. greater than that of A. If a tunnel is to be built through the mountain, at what angle of elevation should it start from A so that it will come out at the level of B?



10. The Empire State Building on 34th St., New York, is 1248 ft. high. One day a man with a large telescope on 42nd St. let people look at the top of the building for 10¢. I noticed that his telescope was set at an angle of elevation of 31°. How far is 34th St. from 42nd St.?

11. A pilot at an altitude of 450 ft. wants to land at an aviation field 3000 ft. distant from his present position. If the safe angle of landing should not exceed 10°, may he descend to the field in a straight line or must he circle it first?

Trigonometry in Athletics (Optional)

When a heavy body leaves the ground, the distance it will go in a horizontal direction is given by the formula

$$d = \frac{v^2 \sin A \sin B}{16}$$

where v is its starting velocity, A is its angle of elevation and $B = 90 - A$.

1. In the broad jump, Paul can start with a velocity of 20 ft. a sec. How far will he jump if he leaves the ground at an angle of elevation of (*a*) 30°? (*b*) 45°? (*c*) 60°? (*d*) At what angle of elevation should he jump to go the farthest?

2. In the Olympic record broad jump, the winner jumped with a velocity of about 28 ft. a sec. How far would he jump if his angle of elevation was (*a*) 30°? (*b*) 45°? (*c*) 65°? (*d*) At which angle do you think he made the record jump, and what was his distance?

3. The world's record holder can throw a 16-lb. shot at a velocity of 41 ft. a sec. How far can he throw it if it leaves his hand at an angle of elevation of (*a*) 20°? (*b*) 45°? (*c*) 70°? (*d*) At which angle must he have thrown the shot when he made his record and how far did he throw it?

4. In the Olympic contests, the winner in the 16-lb. hammer-throwing contest could throw his hammer with a velocity of 76 ft. a sec. If he threw it at the most advantageous angle, what is his record?

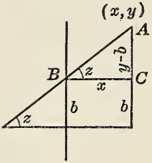
5. Find, by trying several angles, at what angle of elevation a baseball player should try to hit the ball if he is attempting to make a home run.

Do you think that the angle of elevation at which a body will travel farthest is always the same?

The line $y = mx + b$ (*Optional*)

To find the angle made by the line $y = mx + b$ with the x axis, we first solve the equation for m.

$$\begin{aligned} y &= mx + b \\ mx &= y - b \\ m &= \frac{y - b}{x} \end{aligned}$$



From the figure, you can see that $\frac{y - b}{x}$ is $\frac{AC}{BC}$

$$m = \tan z$$

The tangent of the angle made by the line with the x axis is m.

How to find the angle z.

Illustration: Find z for the line $2x - 3y = 5$.

Solving for y: $-3y = -2x + 5$

$$y = \tfrac{2}{3}x - \tfrac{5}{3}$$

$$m = \tan z = \tfrac{2}{3} = .6667$$

From the table: $z = 34°$

Exercises

Find the angle z made by these lines with the x axis:

1. $y = x + 4$
2. $y = 2x - 5$
3. $y = \frac{1}{3}x + 4$
4. $3x - y = 2$
5. $2x - 5y = -4$
6. $y = 3$

INTERPOLATION (Optional)

How to find the functions of angles that are not given in the table. If sin 34° = .5592 and sin 35° = .5736, what is the value of sin 34½° or sin 34°30′? How would it compare with the values for 34° and 35°?

Can you now find sin 34°15′? How much bigger is sin 35° than sin 34°? What part of this difference should correspond to 15′?

Find sin 34°10′. What part of a degree is 10′? What part of the difference between sin 34° and sin 35° should correspond to this?

Illustration. Find sin 52°16′.

60′		sin 53°	= .7986		106
	16′	sin 52°16′	=	d	
		sin 52°	= .7880		

$$\frac{16}{60} = \frac{d}{106}$$

$$60\,d = 1696$$

$$d = 28.3$$

$$\sin 52°16' = .7908$$

The difference in the angles is 60′, and the difference in the sines is 106. (Omit the decimal point.) d should be the same part of 106 as 16′ is of 60′. To the nearest unit, $d = 28$. Since 16′ corresponds to a difference of 28, sin 52°16′ will be 28 more than 7880 or will be 7908.

Verify these functions:

(*a*) sin 8°40′ = .1507

(*b*) tan 27°12′ = .5139

(*c*) sin 37°25′ = .6076

(*d*) sin 83°48′ = .9941

Exercises

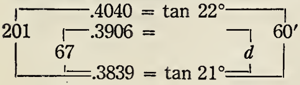
Find the value of:

1. sin 10°15′
2. sin 68°20′
3. sin 44°45′
4. sin 67°40′
5. sin 4°50′
6. tan 8°30′
7. tan 22°45′
8. tan 61°10′
9. tan 33°12′
10. tan 84°6′
11. sin 12°57′
12. tan 69°14′
13. tan 42°28′
14. sin 20°38′
15. tan 46°46′

How to find angles when their functions are not in the table. 1.2349 = tan 51° and 1.2799 = tan 52°. Of what angle is 1.2574 the tangent? How much do the tangents of 51° and 52° differ? How many minutes does this difference correspond to? If tan x were 225 larger than tan 51° (neglecting the decimal point), how many minutes larger than 51° must x be?

Of what angle is .3906 the tangent? What two values of the tangent in the table are nearest to .3906? By how much do they differ? Of what angle is the value just above it the tangent? That just below it? By how much does .3906 exceed the smaller of these tangents? What part of the difference above is this? How many minutes should correspond to it?

Illustration. Find y if tan y = .3906.



$$\frac{67}{201} = \frac{d}{60}$$
$$d = 20$$
$$y = 21°20′$$

In the table, the nearest tangents to .3906 are .3839 and .4040. Their difference is 201. The difference between 3906 and 3839 is 67. d must be the same part of 60′ that 67 is of 201. y is 20′ larger than 21°.

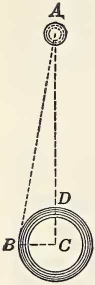
Exercises

Find the value of x if:

1. tan x = .4770
2. tan x = .9111
3. sin x = .4962
4. sin x = .8923
5. sin x = .9629
6. tan x = 2.1875

Trigonometry for Boys and Girls

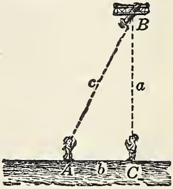
1. Ruth and Mildred found the height of a flag-pole by measuring a distance of 50 ft. from its base and by finding that from that point the angle of elevation of the top was 48°40′. How high was the pole?



2. Allan found that his school cast a shadow 20 ft. long. In an almanac he found that the angle of elevation of the sun at that time was 43°35′. How high was the school?

3. If the radius of the earth is 4000 mi., and if $\angle B$ is 89°4′, how far is the moon from B?

4. When airplane B was directly over C, Paul at A, 1200 ft. from C, found its angle of elevation to be 74°55′. How high was the airplane?

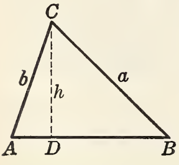


5. From the top of a mountain 6000 ft. high, Sarah found that the angle of depression of a lake was 28°15′. How far was she from the lake?

6. Robert wants to run an aerial from his window to the top of a pole 32 ft. away. He finds the angle of elevation of the top of the pole to be 42°48′. What length of wire will he need?

7. John was told that it was unsafe to climb a ladder unless the angle it made with the ground was less than 65°. He placed a 14-ft. ladder against the house so that the foot of it was 5 ft. 3 in. from the house. Find to the nearest minute the angle it then made with the ground.

Thought Questions



1. In triangle ABC, $a = 10$, $\angle B = 48°$, and $\angle A = 37°$. By using the two right triangles CDA and CDB, find b.

2. Write a general solution, that is, a formula for b in terms of a, B, and A. Can you write a formula for finding h in each triangle and then eliminate h?

3. Using the formula that you have derived, find b when $a = 50$, $\angle A = 66°$, and $\angle B = 48°$.

Review Exercises

1. In triangle ABC, $\angle C = 90°$, $\angle A = 53°$, and $AC = 100$. Find the length of BC to the nearest tenth.

2. In triangle ABC, $\angle C = 90°$, $AB = 50$, $BC = 30$, and $AC = 40$. Express sin A and tan A as decimals.

3. As angle A increases from 0° to 90°, does sin A increase or decrease? Does tan A increase or decrease? Cos A?

4. Find the value of:

(*a*) sin 12° (*c*) sin 82° (*e*) tan 45°
(*b*) tan 67° (*d*) tan 5° (*f*) sin 53°

5. Find the angle whose sine is:

(*a*) .4384 (*b*) .9903 (*c*) .5

6. Find the angle whose tangent is:

(*a*) .0349 (*b*) 9.5144 (*c*) 1

7. Find to the nearest degree, the angle whose:

(*a*) sine is .9801 (*b*) tangent is .1504

8. A wire attached to the top of a telephone pole reaches the ground 10 ft. from the pole and makes an angle of 74° with the ground. How high is the pole?

9. An airplane is 800 ft. above the ground and directly over a school. The angle of elevation of the plane as seen by a boy at a point on the ground some distance from the school is 22°. How far is the boy from the school, and how far from the plane?

10. The ratio of a leg of a right triangle to the hypotenuse is $\frac{7}{8}$. Find to thousandths the sine of the opposite angle. Find the angle to the nearest degree.

11. As an angle increases from 0° to 90°, which increases faster, the sine or the tangent?

12. From the top of a cliff 700 ft. high, Paul observes that the angle of depression of a boat is 42°. How far is the boat from the foot of the cliff?

13. Find the angle of elevation of the sun when a building 50 ft. high casts a shadow 46 ft. long.

14. Find the length of the shadow cast by a 12-ft. pole held vertically when the sun is 28° above the horizon.

15. How long a wire is needed to reach from the top of a pole 30 ft. high to the ground and make an angle of 64° with the ground?

16. A 20-ft. ladder is to be placed against a building so that it will make an angle of 75° with the ground. How far from the building should the foot of the ladder be placed?

17. Do E, F, and G together form a pleasing chord? Show how you know.

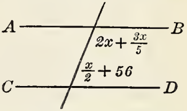
18. Solve algebraically: $5x + 2y = 8$, $4x + 3y = 5$

19. Separate 300 into two parts in the ratio 11:4.

20. A pole 4 ft. tall casts a shadow 5 ft. long at the same time that a building casts a shadow 65 ft. long. How high is the building?

21. Solve: $\frac{5x}{3} - \frac{x-4}{5} = x + 12$

22. Find x if AB is parallel to CD.



23. Find the price goods must be marked, if the cost is \$8.40, a profit of 25% of the cost is to be made, and a discount of 40% of the marked price is to be given.

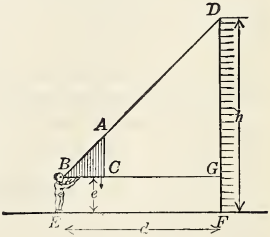
$$\text{Formula: } m = \frac{c(1+p)}{1-d}$$

24. Solve for x: $5x - 2 = 10 + 2x$

25. A grocer has 100 lbs. of 40¢ coffee. How many pounds of 25¢ coffee must he use to make a mixture worth 30¢ a lb.?

26. Using $F = \frac{9C}{5} + 32$, find F when $C = 20$.

27. In the triangle ABC, the length of BC is a and of AC is b. h, d, and e are as shown in the figure. Show that the formula for h is $h = \frac{bd}{a} + e$.



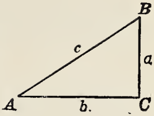
28. If $AC = 5$ in., $BC = 3$ in., $BE = 5$ ft., and $EF = 24$ ft., find the length of DF.

29. If the triangle has an altitude of 9 in. and a base of 3 in. and the boy, whose eye is 5 ft. from the ground, stands 20 ft. from the building, what is the height of the building?

30. The horsepower of an automobile is given by the formula $h = .4\,d^2n$. Find h when $d = 5$ and $n = 8$.

Test

In triangle ABC, $\angle C$ is the right angle. Find all of the parts not given in the table, lines to three significant figures and angles to the nearest degree.



	A	B	a	b	c
1.	40°		24		
2.	56°				100
3.	71°			300	
4.		30°	160		
5.		68°		400	
6.		51°			600
7.			76	100	
8.				89	200
9.			28.7		500
10.	60°				1

Chapter 14

LOGARITHMS

How we can use exponents to save us work. Here is a table of the powers of 2:

$2^1 = 2$	$2^7 = 128$	$2^{13} = 8192$
$2^2 = 4$	$2^8 = 256$	$2^{14} = 16{,}384$
$2^3 = 8$	$2^9 = 512$	$2^{15} = 32{,}768$
$2^4 = 16$	$2^{10} = 1024$	$2^{16} = 65{,}536$
$2^5 = 32$	$2^{11} = 2048$	$2^{17} = 131{,}072$
$2^6 = 64$	$2^{12} = 4096$	$2^{18} = 262{,}144$

By using this table we can do certain computations easily.

Illustration 1. Multiply 8192 by 16.

From the table we see that $8192 = 2^{13}$ and $16 = 2^4$. So:

$$8192 \times 16 = 2^{13} \times 2^4 = 2^{17}$$

Looking again at the table, we see that $2^{17} = 131{,}072$. So:

$$8192 \times 16 = 131{,}072$$

Illustration 2. Divide 65,536 by 2048.

$$65{,}536 \div 2048 = 2^{16} \div 2^{11} = 2^5 = 32$$

Illustration 3. Find the square root of 16,384.

$$\sqrt{16{,}384} = \sqrt{2^{14}} = 2^7 = 128$$

Exercises

Find the value of:

1. 32×512
2. 256×128
3. 64×4096
4. $8 \times 16{,}384$
5. 8192×32
6. $131{,}072 \div 8192$
7. $16{,}384 \div 512$
8. $262{,}144 \div 65{,}536$
9. $4096 \div 256$
10. 128^2
11. $\sqrt{262{,}144}$
12. $\sqrt{4096}$
13. $\sqrt{65{,}536}$
14. $\sqrt[3]{32{,}768}$
15. 16^4

It may occur to you that this table is not of much use, for you may want to multiply numbers that are not in it. Let us see if we can discover a way of using exponents for multiplying or dividing any two numbers. To do this we must study exponents that are not integers.

What is a fractional exponent? Does $x^{\frac{1}{2}}$ mean anything to you? When you multiply numbers, what do you do with their exponents? What is $x^4 \cdot x^5$? $a^8 \cdot a^7$? $x^{\frac{1}{2}} \cdot x^{\frac{1}{2}}$? $b^{\frac{3}{2}} \cdot b^{\frac{3}{2}}$?

What is the value of $\sqrt{9}$? Does this have any relation to the fact that $3 \times 3 = 9$? What is $\sqrt{49}$? Is it related to the fact that $7 \times 7 = 49$?

If the square root of a number is multiplied by itself, what is the result? Find: $\sqrt{25} \cdot \sqrt{25}$, $\sqrt{81} \cdot \sqrt{81}$, $\sqrt{x} \cdot \sqrt{x}$, $\sqrt{abcd} \cdot \sqrt{abcd}$, $\sqrt{894} \cdot \sqrt{894}$.

The square root of a number is one of the two equal factors of the number. $\sqrt{25} = 5$ because $5 \times 5 = 25$.

If $x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = x$, what is the relation of $x^{\frac{1}{2}}$ to x? Does $x^{\frac{1}{2}} = \sqrt{x}$?

What is the value of $9^{\frac{1}{2}}$? $16^{\frac{1}{2}}$? $36^{\frac{1}{2}}$? $81^{\frac{1}{2}}$? $144^{\frac{1}{2}}$?

What is the value of $x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} \cdot x^{\frac{1}{3}}$? If the cube root of x ($\sqrt[3]{x}$) is one of the three equal factors of x, is $x^{\frac{1}{3}}$ the same as $\sqrt[3]{x}$?

Find the value of $8^{\frac{1}{3}}$, $27^{\frac{1}{3}}$, $64^{\frac{1}{3}}$.

What is the value of $a^{\frac{3}{2}} \cdot a^{\frac{3}{2}}$? Express $a^{\frac{3}{2}}$ as a root of a. What does the numerator, 3, tell you? What does the denominator, 2, tell you?

When a quantity has a fractional exponent, the ***numerator*** of the fraction indicates the ***power,*** and the ***denominator*** indicates the ***root.*** For example, $x^{\frac{4}{3}}$ means the cube root of x^4, or $x^{\frac{4}{3}} = \sqrt[3]{x^4}$.

Illustrations:

$$a^{\frac{2}{5}} = \sqrt[5]{a^2}$$
$$9^{\frac{1}{2}} = \sqrt{9} = 3$$
$$16^{\frac{3}{4}} = \sqrt[4]{16^3} = 2^3 = 8$$
$$\sqrt[3]{x^2y} = x^{\frac{2}{3}}y^{\frac{1}{3}}$$
$$(8\,a^3)^{\frac{2}{3}} = (\sqrt[3]{8\,a^3})^2 = (2\,a)^2 = 4\,a^2$$

Exercises

Write these exercises with roots instead of fractional exponents:

1. $x^{\frac{1}{2}}$
2. $a^{\frac{2}{3}}$
3. $m^{\frac{3}{2}}$
4. $v^{\frac{3}{4}}$
5. $m^{\frac{5}{2}}$
6. $r^{\frac{4}{3}}$
7. $t^{\frac{1}{2}}$
8. $A^{\frac{1}{3}}$
9. $3\,x^{\frac{1}{2}}$
10. $4\,K^{\frac{1}{3}}$
11. $7\,D^{\frac{2}{5}}$
12. $5\,x^{\frac{2}{3}}$

Write these exercises with fractional exponents instead of roots:

13. $\sqrt{a}$
14. $\sqrt{m^3}$
15. $\sqrt{x^5}$
16. $\sqrt{xy}$
17. $\sqrt[3]{b}$
18. $\sqrt[3]{M}$
19. $\sqrt[4]{y^3}$
20. $\sqrt[3]{y^4}$
21. $2\,\sqrt{t}$
22. $a\,\sqrt{b}$
23. $5\,\sqrt{x^3}$
24. $\sqrt{2\,ab}$

Illustration. Find the value of $8^{\frac{2}{3}}$:

$$8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = 2^2 = 4$$

Find the value of:

25. $4^{\frac{1}{2}}$
26. $16^{\frac{1}{2}}$
27. $25^{\frac{1}{2}}$
28. $64^{\frac{1}{2}}$
29. $4^{\frac{3}{2}}$
30. $16^{\frac{1}{4}}$
31. $8^{\frac{1}{3}}$
32. $25^{\frac{3}{2}}$
33. $1^{\frac{1}{2}}$
34. $1^{\frac{2}{3}}$
35. $27^{\frac{1}{3}}$
36. $32^{\frac{2}{5}}$

Find the value to three significant figures of:

37. $2^{\frac{1}{2}}$
38. $7^{\frac{1}{2}}$
39. $355^{\frac{1}{2}}$
40. $89.47^{\frac{1}{2}}$
41. $5^{\frac{3}{2}}$
42. $2^{\frac{5}{2}}$

The zero exponent. Find the value of $\frac{x^7}{x^4}$, of $\frac{x^9}{x^2}$, of $\frac{x^{13}}{x^5}$. What did you do to the exponents to get your answer? When you divide one power of a letter by another power of that letter, do you always subtract exponents? Then what exponent has x in the answer to $\frac{x^7}{x^7}$? But what is the value of $\frac{x^7}{x^7}$, that is, of any number divided by itself?

Any quantity to the zero power equals 1.

Sometimes 0^0 is an exception.

By the law of exponents: $\frac{x^3}{x^3} = x^{3-3} = x^0$

But by ordinary division: $\frac{x^3}{x^3} = 1$

Consequently: $x^0 = 1$

Exercises

1. What is the value of: a^0? $(2b)^0$? $(x+1)^0$? $(x+y)^0$? $x^0 + y^0$? $3m^0$? 2^0? 10^0?

2. What power of 10 is 1?

3. Express as powers of 10: 100; 10,000; 1; 1,000,000; 10; 1000

4. One followed by 26 zeros is what power of 10?

5. $26{,}000{,}000 = 2.6 \times 10{,}000{,}000 = 2.6 \times 10^7$. Express in the same way, that is, with but one digit on the left of the decimal point: 38,000,000; 74,000; 6,500,000; 24

6. How many thousand is 9840? How many hundred thousand is 864,300? Express the latter using a power of 10 for the hundred thousand.

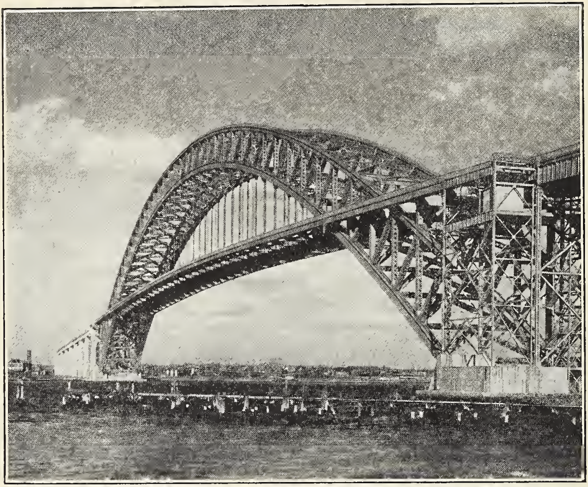
7. Express these numbers as numbers between 1 and 10 multiplied by a power of 10:

(*a*) 8000
(*b*) 40,000
(*c*) 7600
(*d*) 834,000
(*e*) 62,840
(*f*) 1,960,000
(*g*) 53,400
(*h*) 3
(*i*) 81,659

8. What is the value of: 1^3? 1^7? 1^{105}? $1^{\frac{1}{2}}$? $1^{\frac{3}{8}}$? 1^0?

Express 2 as a power of 8. Since $2 \times 2 \times 2 = 8$, $2 = 8^{\frac{1}{3}}$. We could write this exponent as a decimal by changing $\frac{1}{3}$ to .3333, so $2 = 8^{.3333}$.

You have just seen that numbers which are not exact powers of a number can be expressed as decimal powers of the number. A little over 300 yrs. ago, a Scotchman named John Napier discovered that all numbers could be expressed as powers of a single number. He called these exponents *logarithms*. With the help of another man, Briggs, he made a table by means of which you can change



Courtesy Port of New York Authority.

MATHEMATICS AT WORK

All construction work involves a great deal of mathematical computation.

all multiplication exercises into easy addition exercises, and all division exercises into subtraction exercises. This table expresses all numbers as powers of 10. We shall now learn how to use it.

How to use a table of logarithms. The table of logarithms is found on pages 344 and 345. These logarithms are decimals, but as all decimal points have been omitted from the table, you will have to put them in yourself. The first two digits of the number are found in the column headed *N*, and the third digit is found across the top of the page. We shall understand that there is a decimal point after the first digit. The rest of the page gives the exponents or logarithms, and in these we shall understand that there is a decimal point before the whole number.

Illustration. Find the logarithm of 3.87.

As the table contains no decimal points, we shall look first for 38 on the left. Then we look across that line until we come to the column headed with the third digit, 7. There we find 5877. Now put the decimal point before this, and we have $3.87 = 10^{.5877}$. (Do not forget that these logarithms are exponents of 10.)

Exercises

Here are numbers with their logarithms. Look them up in the table to see if all of them are correct:

1. $2.5 = 10^{.3979}$
2. $5.4 = 10^{.7324}$
3. $7.2 = 10^{.8573}$
4. $9 = 10^{.9542}$
5. $3 = 10^{.4771}$
6. $1.42 = 10^{.1523}$
7. $8.91 = 13^{.9499}$
8. $3.65 = 10^{.5623}$
9. $7.28 = 10^{.8621}$
10. $4.59 = 10^{.6618}$

Write the logarithms of:

11. 2.8
12. 7.6
13. 4.3
14. 8.5
15. 1.2
16. 5.0
17. 2.0
18. 4
19. 9
20. 6
21. 2.41
22. 3.82
23. 7.84
24. 9.68
25. 6.52
26. 8.66
27. 3.33
28. 2.01
29. 1.11
30. 4.99

Write the number that corresponds to:

31. $10^{.5185}$
32. $10^{.6812}$
33. $10^{.2788}$
34. $10^{.3032}$
35. $10^{.4393}$
36. $10^{.8457}$
37. $10^{.6590}$
38. $10^{.8797}$
39. $10^{.8488}$
40. $10^{.9805}$
41. $10^{.9996}$
42. $10^{.0043}$
43. $10^{.8500}$
44. $10^{.7642}$
45. $10^{.0000}$

How to use logarithms in computation. When you multiply numbers, what do you do with the exponents? When you divide numbers, what do you do with the exponents? What should you do with their logarithms, when you multiply numbers? What should you do with their logarithms, when you divide numbers?

COMMON LOGARITHMS OF NUMBERS

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
N	0	1	2	3	4	5	6	7	8	9

COMMON LOGARITHMS OF NUMBERS (*Continued*)

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9

Illustration 1. Multiply 3.12 by 2.47.

$$\begin{aligned} 3.12 &= 10^{.4942} \\ 2.47 &= 10^{.3927} \\ 3.12 \times 2.47 &= 10^{.4942 + .3927} \\ &= 10^{.8869} \end{aligned}$$

The nearest logarithm to this in the table is 8871, and this corresponds to 7.71, so:

$$3.12 \times 2.47 = 7.71$$

Illustration 2. Find the value of 6.84 ÷ 4.73.

$$\begin{aligned} 6.84 &= 10^{.8351} \\ 4.73 &= 10^{.6749} \\ 6.84 \div 4.73 &= 10^{.8351 - .6749} \\ &= 10^{.1602} \end{aligned}$$

The nearest logarithm to 1602 is 1614, and this corresponds to 1.45. So:

$$6.84 \div 4.73 = 1.45$$

Illustration 3. Find the square root of 3.29.

$$\begin{aligned} \sqrt{3.29} &= \sqrt{10^{.5172}} \\ &= 10^{.2586} \\ &= 1.81 \end{aligned}$$

Exercises

Find the value of:

1. 2.84×3.17
2. 8.34×1.18
3. 6.45×1.32
4. 2.98×3.16
5. 7.79×1.15
6. 2.32×3.41
7. 3.64×1.72
8. 4.68×2.12
9. 1.94×3.85
10. $8.79 \div 3.64$
11. $4.53 \div 3.68$
12. $5.58 \div 4.97$
13. $8.96 \div 8.14$
14. $6.71 \div 4.38$
15. $7.82 \div 3.69$
16. $5.91 \div 4.15$
17. $3.38 \div 2.74$
18. $1.76 \div 1.51$
19. $\sqrt{7.37}$
20. $\sqrt{5.44}$
21. $\sqrt{9.98}$
22. 2.07^3
23. 1.86^2
24. $\sqrt[3]{6.92}$
25. $\sqrt[3]{3.35}$
26. $\sqrt[5]{7.69}$
27. $\sqrt[4]{2.63}$

In these exercises the decimal point came after the first digit. But suppose you wanted to multiply numbers in which it occurred somewhere else. What would you do then? Let us examine such a number.

Illustration 1. Find the logarithm of 47.8.

47.8 is 10 times as large as 4.78.

$$\begin{aligned} 4.78 &= 10^{.6794} \\ 10 &= 10^{1} \\ 47.8 &= 10^{.6794+1} \\ &= 10^{1.6794} \end{aligned}$$

Illustration 2. Find the logarithm of .00736.

.00736 is 7.36 ÷ 1000 or by 10^3.

$$\begin{aligned} 7.36 &= 10^{.8669} \\ 1000 &= 10^{3} \\ .00736 &= 10^{.8669-3} \end{aligned}$$

Notice that now our logarithm is made up of two parts, a whole number and a decimal. The whole number is called the ***characteristic,*** and the decimal is called the ***mantissa.***

How to find the characteristic. You have already learned to find the mantissa in the table. Now let us examine the illustration above to see how we could find the characteristic.

When the decimal point was after the first digit, as in 4.78, the characteristic was 0. When we moved the decimal point one place to the right, as in 47.8, we multiplied the number by 10, and this made the characteristic 1. If we moved the decimal point another place to the right, it would again multiply the number by 10 and so add another 1 to the characteristic. So you see then that for every place the decimal point is moved to the right, the characteristic is increased by 1. To find the characteristic, it is only necessary to count the number of places the decimal point is moved to the right.

In the second illustration, the decimal point in .00736 is three places farther to the left than in 7.36. This divides the number by 10^3 and so subtracts 3 from the logarithm. Therefore, for every place the decimal point is moved to the left, we subtract 1 from the logarithm.

To find the characteristic, first think of the decimal point as being after the first significant figure of the number. Then count the number of places from this imaginary decimal point to the real decimal point. If you count to the right, add that number to the logarithm. If you count to the left, subtract that number from the logarithm.

Illustration 1. Find the characteristic of the logarithm of 384. If the decimal point were after the 3, the characteristic would be 0. Since it is two places to the **right,** the characteristic is 2.

Illustration 2. Find the characteristic of the logarithm of .061. If the decimal point were after the 6, the characteristic would be 0. Since it is two places to the **left,** the characteristic is -2.

Illustration 3. Find the characteristic of the logarithm of 7.35. If the decimal point were after the 7, the characteristic would be 0. Since it is there, the characteristic is 0.

When you know the characteristic of a logarithm, you can use this same rule reversed for finding the decimal point in the number.

Illustration 4. The characteristic of the logarithm for the set of digits 942 is 5. Put in the decimal point. If the characteristic had been 0, the decimal point would have been after the 9. Since it is 5, we count five places to the **right,** adding 0's when needed, and the number is 942000.

Illustration 5. The characteristic of the logarithm for the set of digits 346 is -4. Put in the decimal point. If the characteristic had been 0, the decimal point would have been after the 3. Since it is -4, we count four places to the **left,** adding 0's when needed, and the number is .000346.

Exercises

Give the characteristic of the logarithm of:

1. 6
2. 5.8
3. 7.94
4. 84
5. 63.1
6. 125
7. .7
8. .34
9. .828
10. .0045
11. .0354
12. .0001

Using the set of digits 6452, put in a decimal point for each of these characteristics:

13. 0
14. 2
15. -3
16. 1
17. -5
18. 6
19. 3
20. -1

You have now worked with logarithms long enough so that you know that they are exponents of 10. From now on, we shall omit the 10 and write the work in a more condensed form. We shall use "log" to stand for "the logarithm of." For example, log 2 = .3010 means "the logarithm of 2 = .3010."

Illustration. Find N if $N = 7420 \times .0653$.

$$\text{Log } N = \log 7420 + \log .0653$$

Before looking up the logarithms in the table, make a complete form for your work:

$$\begin{array}{rl} \log\ 7420 & = \\ + \log .0653 & = \underline{\hspace{3cm}} \\ \log\ N & = \\ N & = \end{array}$$

Then use a table to fill in the blanks, and complete the work:

$$\begin{array}{rl} \log\ 7420 & = 3.8704 \\ + \log .0653 & = \underline{\ .8149 - 2^{*}} \\ \log\ N & = 4.6853 - 2 \\ & = 2.6853 \\ N & = 485 \end{array}$$

Find the value of:

21. 34×5.3
22. 87×45.5
23. $42.3 \div 1.58$
24. $5.93 \times .0042$
25. $85.7 \div 3.82$
26. 3600×6.52
27. $.047 \div .293$
28. $.0569 \div 842$
29. 546^3
30. $\sqrt{76}$
31. $\sqrt{45{,}600}$
32. $\sqrt[3]{72.4}$

---

* When the characteristic is negative, it is customary to change it to a number -10. -2 becomes $8 - 10$, and the logarithm $.8149 - 2$ is written $8.8149 - 10$.

Logarithms in Business

You have already found compound interest by means of a compound-interest table. This was very easy, but you could solve exercises for the rates and times given in the table only. By means of logarithms you can find interest for any rate and any time.

The compound-interest formula is $A = P(1 + r)^n$.

Illustration. Find the amount of \$347 at 4% for 8 yrs.

$$\begin{aligned} A &= P(1 + r)^n \\ &= 347(1 + .04)^8 \\ &= 347(1.04)^8 \\ \log A &= \log 347 + 8 \log 1.04 \end{aligned}$$

$$\begin{array}{rl} \log 1.04 &= .0170 \\ \hline \log 347 &= 2.5403 \\ + \ 8 \log 1.04 &= .1360 \\ \hline \log A &= 2.6763 \\ A &= 475 \end{array}$$

If interest is compounded annually, find the amount of:

1. \$400 at 6% for 10 yrs.
2. \$760 at 4% for 5 yrs.
3. \$645 at 8% for 8 yrs.
4. \$192 at 2% for 6 yrs.
5. \$873 at 6% for 9 yrs.
6. \$84.50 at 4% for 5 yrs.
7. \$1340 at 6% for 3 yrs.
8. \$11,600 at 2% for 4 yrs.
9. \$94.60 at 4% for 25 yrs.
10. \$2630 at 6% for 100 yrs.

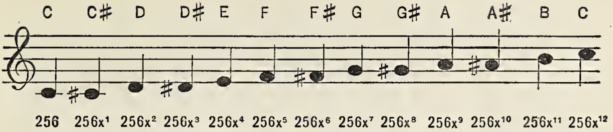
11. In 1623 the Indians sold Manhattan Island for \$24. If they had placed that money at 6% interest, compounded annually, how much would it have amounted to in 1923?

12. If every person who landed at Plymouth from the *Mayflower* in 1620 had 3 descendants and each of these averaged 3 descendants in 25 yrs., etc., down to the present, the number of persons who might have had an ancestor on the *Mayflower* would be more than $3^{12} \times 10^2$. How many people does this represent? How does it compare with the population of the United States in 1920 (105,000,000)?

Logarithms in Music

(Optional)

In planning the tempered scale for the piano, Bach found it necessary to make all ratios the same. Let x be the ratio of any two consecutive half tones in the chromatic scale. Then if C = 256, C♯ = $256\,x$, D = $256\,x^2$, D♯ = $256\,x^3$, and so on to the next C which would be $256\,x^{12}$.



But the next C is double the C below or 512. So $256\,x^{12} = 512$, and $x^{12} = 2$.

1. Using logarithms, find x.

2. Find D on this modified scale, and determine by how many vibrations it differs from the natural D, 288.

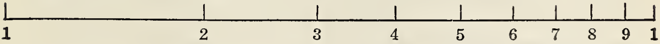
3. Find G on the modified scale, and determine how much it differs from the natural G, 384.

THE SLIDE RULE

(Optional)

Engineers and scientists multiply, divide, extract square root, find sines, tangents, and logarithms, solve ratio and proportion; and perform other computations on a ruler called a *slide rule*. By simply pulling a slide out the proper distance, the answer is found without the trouble of figuring.

How the slide rule is made. Let us draw a line 10 in. long and lay off logarithms on it. From our table, we find that log 1.0 is .0000, so we shall put 1 at the end of the rule.

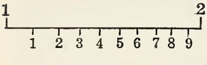


Log 2.0 is .3010, so we shall take .3010 × 10 in. or about 3 in. as the length to represent 2. Log 3 is .4771, so we take .48 × 10 in. or 4.8 in. to represent 3.

What length shall we take to represent 4? 5? 6? 7? 8? 9?

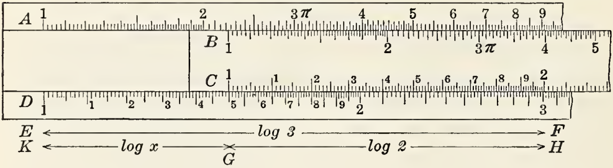
We can also put on the second figure of the number. For example, log 1.1 = .0404; therefore, 1.1 would be placed about .04 × 10 in. or .4 in. from the end.

Where would you put 1.2? 1.3? 1.4? 1.5? Here is the way the section from 1 to 2 looks when we put on the second digit. In the same way we could divide these parts to show the third digit. Only a part of this third digit is on the slide rule as the marks would be too close to each other at the right end of the rule.



The slide rule has two scales, C and D, like those we have just made. They are exactly alike, but D is on the body of the rule, whereas C is on the slide and can be made to fit on D.

How to divide on the slide rule. Let us divide 3 by 2. You remember that to divide by logarithms, you subtract the logarithms. So here we must subtract lengths. Find 3 on scale D, and put the line of the glass on it to mark the place. Now move the slide until the 2 on scale C is under the line of the glass. Then the length *EF* represents log 3 and the length *GH* represents log 2. Then log 3 − log 2 is *EF* − *GH* or *KG*. The answer is the number on scale D where scale C ends. Here the end of scale C is between the first digits 1 and 2, and at the second digit 5. The answer is 1.5. Then 3 ÷ 2 = 1.5.



On the slide rule, we disregard the decimal point; then, the same position can represent 1.5, 15, 150, or .15. You can place the decimal point by estimating your answer. Also, since the decimal point is not used, the 10 and the 1 are the same; so if the 1 is off the scale, find the answer under the 10 at the other end of the slide.

Class Exercises

1. 8 ÷ 2
2. 9 ÷ 3
3. 6 ÷ 3
4. 7 ÷ 2
5. 9 ÷ 6
6. 12 ÷ 4
7. 18 ÷ 6
8. 35 ÷ 7
9. 56 ÷ 4
10. 84 ÷ 6
11. 58 ÷ 18
12. 94 ÷ 38
13. 75 ÷ 41
14. 28 ÷ 56
15. 39 ÷ 65
16. 136 ÷ 57
17. 192 ÷ 43
18. 117 ÷ 24
19. 184 ÷ 67
20. 108 ÷ 95

Optional Exercises

21. 364 ÷ 435
22. 298 ÷ 342
23. 655 ÷ 276
24. 98.5 ÷ 33
25. 36.2 ÷ 47
26. 1.83 ÷ 39.2
27. 1.37 ÷ 4.95
28. 12.2 ÷ 184
29. .149 ÷ 1.23
30. 6.85 ÷ .735
31. 1.01 ÷ 11.1
32. 38.8 ÷ 2.46
33. 4.75 ÷ .011
34. .745 ÷ .835
35. 555 ÷ .0146

Honor Work

36. 347 ÷ 683
37. 592 ÷ 837
38. 764 ÷ 233
39. 531 ÷ 898
40. 265 ÷ 219
41. 4.84 ÷ 76.3
42. 87.7 ÷ 8.14
43. .0504 ÷ .02
44. 3.59 ÷ .979
45. .999 ÷ .777
46. 1475 ÷ 1235
47. 1925 ÷ 1147
48. 1032 ÷ 2235
49. 2045 ÷ 1468
50. 1273 ÷ 1457

How to multiply on the slide rule. Remember that in multiplication we add logarithms. So on the slide rule we must add lengths that represent logarithms.

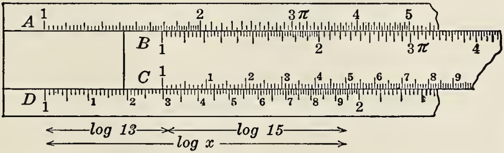


Illustration. Multiply 13 by 15.

Place the line on the glass over the 13 on the D scale. Now move the slide until one end of the C scale is under the line on the glass. Leave the slide in that position, and move the glass until the line on it is over the 15 on the C scale. The answer, where the line on the glass crosses the D scale, is 195.

If the answer is off the D scale, use the other end of the C scale under the first position of the glass.

Class Exercises

1. 12×6
2. 23×3
3. 14×6
4. 18×4
5. 26×3
6. 41×32
7. 28×76
8. 82×17
9. 32×26
10. 28×19
11. 23.6×41.5
12. 4.95×1470
13. $.695 \times 3.47$
14. $1875 \times .226$
15. 48.3×88.7

Illustration. Find the value of $\frac{264 \times 7.35}{473}$.

Set 473 on scale C at 264 on scale D. Then set the line of the glass at 735 on scale C. 41 under the line on scale D is the answer. Approximate the answer to locate the decimal point. $\frac{200 \times 7}{400} = 3.5$. There is 1 digit left of the decimal point, and the answer is 4.1.

Optional Exercises

Find the value of:

16. $\frac{38.2 \times 47.6}{805}$
17. $\frac{.0465 \times 278}{1.49}$
18. $\frac{8.65 \times 3.95}{47.5}$
19. $\frac{26.8 \times .294}{65.5}$
20. $\frac{.0042 \times .057}{.000835}$
21. $\frac{384 \times 1.04}{26.7}$

Honor Work

22. $\frac{5.02^2}{82.9}$
23. $\frac{3840^2}{.01^2}$
24. $\frac{47^2 \times 8.91}{392^2}$

Solve for x:

25. $81.7\,x = 384$
26. $.246\,x = 8.95$
27. $6.35\,x - 2.4 = 8.85$
28. $39.7\,x = 475 - 3\,x$
29. $\frac{x}{58.3} = \frac{7.92}{.675}$

How to find square root on the slide rule. Let us examine scale A. The numbers run from 1 to 9 and then start over again. The 1 in the middle of the scale is really a 10. Similarly the numbers to its right are 20, 30, 40, etc. That is, add a 0 after each. These numbers on scale A are the squares of the numbers directly below them on scale D. Consequently, to find square root, it is only necessary to set the glass over the number on scale A and read the answer on scale D.

You remember that when we found square root, we first divided the number up into groups of two figures each beginning at the decimal point. Now if the first group contains but one digit—that is, is less than 10—locate the number on the left half of the A scale. If it contains two figures—that is, is between 10 and 100—locate it on the right half of A.

Exercises

Check these answers on your rule:

1. $\sqrt{735} = 27.1$ **2.** $\sqrt{2140} = 46.3$ **3.** $\sqrt{.00296} = .0544$

Find the square root of:

4. 9
5. 16
6. 441
7. 8670
8. 3.85
9. 47.8
10. .0182
11. .246
12. 1.87
13. 562
14. 8.77
15. .305

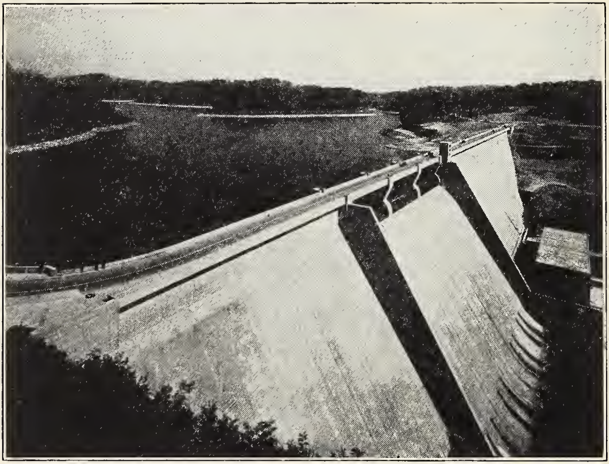
The Slide Rule in Science and Engineering

Scientists solve many problems by means of the slide rule.

Illustration. The time required for the swing of a pendulum is given by the formula $t = \pi\sqrt{\frac{l}{g}}$ where l is the length of the pendulum and g is the acceleration of gravity. Find t when $l = 4.83$ and $g = 32.4$.

Here $t = 3.14\sqrt{\frac{4.83}{32.4}}$. This problem can be solved by a single setting of the slide. Compute the part under the radical on scales A and B, then the part outside on scales C and D. Set the glass at 4.83 on scale A. Move the slide until 32.4 of scale B is under the glass. Move the glass to 3.14 on scale C, and read the answer

1.21 on scale D. For ease in computation, compare this with arithmetic and even with logarithms.



Courtesy of Tennessee Valley Authority.

HARNESSED WATER FOR LIGHT AND POWER

Engineers engaged in the construction of the Wilson Dam often used the slide rule to simplify computation.

Exercises

1. The velocity needed to produce a certain amount of energy is $v = \sqrt{\frac{2k}{m}}$. Find v when $k = 84$ and $m = 2.05$.

2. The electric current needed to produce a certain amount of heat is $C = \sqrt{\frac{H}{rt}}$. Find C when $H = 540$, $r = 8.3$, and $t = 6$.

3. The time it takes a weight to fall from an airplane is $t = \frac{1}{4}\sqrt{h}$. Find t when $h = 1800$.

4. The velocity at which a wheel will fly to pieces can be found from the formula $v = \sqrt{\frac{fr}{m}}$, where f is the force it can stand, r its radius, and m the weight of the wheel. Find v when $f = 8000$, $r = 2.6$, and $m = 340$.

5. The horse-power of a racing-boat engine is found by using the formula $h = \frac{d^2n}{13}$, where d is the diameter of the cylinder and n the number of cylinders. Find h when $d = 12$ and $n = 8$.

6. The force exerted by the governor of a steam engine is $f = \frac{wr}{h}$. Find f when $w = 84$ lbs., $r = 7$ in., and $h = 11$ in.

7. The mechanical equivalent of heat is $w = 772\ H$. Find w when $H = 123$.

8. The safe steam pressure in a boiler is found by using the formula $P = \frac{ste}{rf}$. Find P if $s = 60{,}000$, $e = .82$, $t = .317$, $r = 25$, and $f = 5$.

9. The force between two electric charges is $F = \frac{ce_1e_2}{kr^2}$. Find F when $c = 18$, $e_1 = 34$, $e_2 = 73$, $k = 14.2$, and $r = 6.3$.

10. The force required to keep a body moving in a circle is $f = \frac{mv^2}{r}$. Find f when $m = 68$, $v = 3.7$, and $r = 12.6$.

Review Exercises

1. What is the logarithm of 100? Of 1000? Of 10?

2. What is a logarithm?

3. What do we do with their logarithms when we:

(*a*) Multiply numbers?
(*b*) Divide numbers?
(*c*) Square a number?
(*d*) Cube a number?
(*e*) Find the square root of a number?

4. If the logarithm of 2 is .3010, what is the logarithm of 20? Of 200? Of .02?

5. What is the characteristic of the logarithm of 842? Of 8.42? Of 8,420,000? Of .00842?

6. Find the value of:

(*a*) $4^{\frac{1}{2}}$
(*b*) $9^{\frac{1}{2}}$
(*c*) $8^{\frac{1}{3}}$
(*d*) $16^{\frac{3}{4}}$

7. Find the logarithms of:

(*a*) 3.89
(*b*) 64.2
(*c*) .0891
(*d*) 61200
(*e*) 93.2
(*f*) .468
(*g*) 1,000,000
(*h*) .0000001
(*i*) .0404

8. Find the number whose logarithm is:

(*a*) .4771
(*b*) .5502
(*c*) 1.9042
(*d*) 3.5378
(*e*) .9805 − 2
(*f*) .3444 − 1

9. By using logarithms, find the answers, and check by slide rule:

(*a*) 384×695

(*b*) $\frac{54.7}{8.96}$

(*c*) $\sqrt{583}$

(*d*) $\sqrt{58.3}$

(*e*) $7640\sqrt{3920}$

(*f*) $\frac{64 \times 3.15}{41.7}$

10. If $\log x = \log 3 + \log 8$, find x without using the tables.

Test on Logarithms

Find the results by logarithms:

1. 25.3×8.19

2. $.0742 \times 6530$

3. $954 \div 34.5$

4. $\sqrt{67.3}$

5. $5710 \times 3.52 \times .0097$

6. $458\sqrt{3.62}$

7. $\sqrt[3]{75.6}$

8. $.0043 \times 7.26 \div 345$

Chapter 15

POLYNOMIALS

A POLYNOMIAL is an expression containing two or more terms. A polynomial having just two terms is a ***binomial,*** one having just three terms is a ***trinomial.***

ADDITION OF POLYNOMIALS

To add polynomials, write like terms in columns, and add only the like terms. Check by substituting a number for each letter.

Illustration. Add and check, letting $x = 2$ and $y = 1$.

$$\begin{array}{rrrl}
3x^2 & -\,2xy & -\,4y^2 & = \quad 12 - 4 - 4 = 4 \\
-\,2x^2 & +\,3xy & +\,y^2 & = \;\, -8 + 6 + 1 = -1 \\
4x^2 & -\,2xy & +\,5y^2 & = \quad 16 - 4 + 5 = \underline{17} \\
\cline{1-3}
5x^2 & -\,xy & +\,2y^2 & = 20 - 2 + 2 = 20 \longleftrightarrow 20
\end{array}$$

Class Exercises

Add:

1.
$$\begin{array}{rrr}
3a & -\,4b & -\,5c \\
-\,3a & +\,6b & +\,2c \\
a & -\,3b & +\,4c \\
\hline
\end{array}$$

2.
$$\begin{array}{rrr}
5P & +\,2Q & +\,3R \\
-\,3P & -\,2Q & -\,R \\
-\,4P & -\,Q & +\,2R \\
\hline
\end{array}$$

3.
$$\begin{array}{rrr}
-\,2x & +\,3y & -\,z \\
+\,3x & -\,5y & +\,2z \\
-\,x & -\,2y & +\,3z \\
\hline
\end{array}$$

4.
$$\begin{array}{rrr}
x^2 & +\,x & +\,1 \\
-\,2x^2 & +\,3x & -\,7 \\
3x^2 & -\,8x & +\,5 \\
\hline
\end{array}$$

5.
$$\begin{array}{rrr}
4a^2 & -\,3ab & -\,2b^2 \\
-\,2a^2 & -\,ab & -\,5b^2 \\
a^2 & +\,5ab & -\,6b^2 \\
\hline
\end{array}$$

6.
$$\begin{array}{rrr}
x^2 & -\,xy & +\,4y^2 \\
 & -\,2xy & +\,y^2 \\
3x^2 & & -\,5y^2 \\
\hline
\end{array}$$

7.
$$\begin{array}{rrr}
3x_1 & -\,4x_2 & +\,7x_3 \\
-\,5x_1 & +\,9x_2 & -\,6x_3 \\
7x_1 & -\,6x_2 & -\,5x_3 \\
\hline
\end{array}$$

8.
$$\begin{array}{rrr}
P & & +\,R \\
 & -\,3Q & -\,2R \\
2P & +\,Q & \\
\hline
\end{array}$$

9.
$$\begin{array}{r} 3b + 4w - 5h \\ 2b - 3w + 3h \\ -5b + w - h \\ +2b - 3w + 4h \\ \hline \end{array}$$

10.
$$\begin{array}{r} x^3 - 2x^2y - 3xy^2 \\ -x^3 + 2x^2y + 3xy^2 \\ -2x^3 + 4x^2y + xy^2 \\ +3x^3 - x^2y - 5xy^2 \\ \hline \end{array}$$

11.
$$\begin{array}{r} 2ab - 4ac - 3bc \\ ab + 3ac - bc \\ 2ac - bc \\ -4ab + 5ac \\ \hline \end{array}$$

12.
$$\begin{array}{r} 5m - 2p + w \\ 3p - 2w \\ 4m + p + w \\ -6m - 2p \\ \hline \end{array}$$

Optional Exercises

Put like terms in columns. Usually it is best to arrange with the highest exponent first and then run down the powers, or to arrange letters in alphabetical order. Then add:

13. $3a - 2b - c,\ 2a + 3b - 2c,\ 2b - 4a + 3c$
14. $a^2 - 2ab + b^2,\ a^2 - 2b^2 + ab,\ +b^2 + ab - 3a^2$
15. $4x^2 - 5x + 7,\ +3 - 2x^2 + x,\ 2x - 5x^2 - 3$
16. $x^2 + y^2 + z^2,\ x^2 - z^2 - 2y^2,\ y^2 - x^2 + 4z^2$
17. $kl - lm + mn,\ lm - mn + kl,\ mn - kl + lm$
18. $3r - 7q + 5p,\ -3p + 4q - 6r,\ -2r + q - p$

Honor Work

19. $3.2x - 2.5y - 4.3z,\ 1.7x + 1.4y - 3z,\ -x + y + 5.2z$
20. $.02x - .001y - .5z,\ .3x + .01y - z,\ -x - .008y + 2z$
21. $.3a + .4b - 1.6c,\ -2.3a - b + c,\ -.4a + .3b + 1.2c$
22. $ax + by + cz,\ bx - by + z,\ -ax + y - bz$
23. $(a + 3b)x + (a - b)y,\ (a - 2b)x + (-2a + 3b)y$
24. $2\frac{1}{2}x - 1\frac{1}{3}y + 1\frac{1}{4}z,\ -\frac{3}{4}x + 2\frac{1}{6}y - \frac{3}{4}z,\ x - \frac{1}{2}y$
25. $5x^a - 2x^b - 3x^c,\ -4x^a + 3x^b + x^c,\ 3x^a - 3x^b + 4x^c$

SUBTRACTION OF POLYNOMIALS

Subtraction is like addition except that in every term subtracted, we think of the sign as changed. We can check as in addition by substituting numbers for the letters, or we can add the quantity subtracted to the answer.

Illustration: Subtract, and check, letting $a = 3$:

$$\begin{array}{rl} 5a^2 - 7a + 4 & = 45 - 21 + 4 = 28 \\ \underline{3a^2 + 2a - 12} & = 27 + 6 - 12 = \underline{21} \\ 2a^2 - 9a + 16 & = 18 - 27 + 16 = 7 \longleftrightarrow 7 \end{array}$$

Class Exercises

Subtract:

1.
$$\begin{array}{rrr} yr. & mo. & da. \\ 1938 & 8 & 21 \\ \underline{1933} & \underline{5} & \underline{12} \end{array}$$

2.
$$\begin{array}{r} 3\ 8\ 5 \\ \underline{2\ 6\ 3} \end{array}$$

3.
$$\begin{array}{r} 3t^2 + 8t + 5 \\ \underline{2t^2 + 6t + 3} \end{array}$$

4.
$$\begin{array}{r} 5a - 3b + 6 \\ \underline{3a - 4b + 7} \end{array}$$

5.
$$\begin{array}{r} 2K - 3L - M \\ \underline{8K - 2L + M} \end{array}$$

6.
$$\begin{array}{r} 3x^2 - 4x + 7 \\ \underline{-x^2 + 2x + 3} \end{array}$$

7.
$$\begin{array}{rrr} 5 \text{ cwt.} & 18 \text{ lbs.} & 12 \text{ oz.} \\ \underline{3 \text{ cwt.}} & \underline{9 \text{ lbs.}} & \underline{7 \text{ oz.}} \end{array}$$

8.
$$\begin{array}{r} 5c + 18n + 12z \\ \underline{3c + 9n + 7z} \end{array}$$

9.
$$\begin{array}{r} 3x^2 - 4xy - 4y^2 \\ \underline{2x^2 + 3xy - y^2} \end{array}$$

10.
$$\begin{array}{r} 3\ 8\ 6\ 7 \\ \underline{4\ 0\ 5} \end{array}$$

11.
$$\begin{array}{r} 3t^3 + 8t^2 + 6t + 7 \\ \underline{4t^2 \qquad\quad + 5} \end{array}$$

12.
$$\begin{array}{r} x^3 - 2x^2y \qquad\quad + 7y^3 \\ \underline{2x^3 - 3x^2y + xy^2 - y^3} \end{array}$$

13.
$$\begin{array}{r} a + b \\ \underline{-b + c} \end{array}$$

14.
$$\begin{array}{r} a \\ \underline{2a + 3b - 4c} \end{array}$$

15.
$$\begin{array}{r} 0 \\ \underline{3x^2 - 2y - z} \end{array}$$

Optional Exercises

16. From $7a + b + 5c$ take $2a - 3b - 4c$.

17. From $-3a^2 + 9ab + 7b^2$ take $-ab + 2b^2 - 6a^2$.

18. From $4x^2 + 5xy - 6y^2$ take $-6x^2 + 10xy - 8y^2$.

19. From $6x - 4y + 2z$ take $2x - 3y - 5z$.

20. Subtract $4A - 5B + 6C$ from $A - 2B - 4C$.

21. Subtract $5x_1 - 2x_2 - 3x_3$ from $x_1 + 4x_2 + x_3$.

22. Subtract $-2a - b - 4$ from $5a + 4b - 1$.

23. From 0 take $4x^2 - 5xy - 3y^2$.

24. From $2a + d$ take $5a - 3b - c$.

Subtract:

25. $\begin{array}{r} 3.8\,x + 7.34\,y \\ -\,2.2\,x + 3.21\,y \\ \hline \end{array}$

26. $\begin{array}{r} 2\sqrt{x} + 3\sqrt{x-y} \\ \sqrt{x} - 4\sqrt{x-y} \\ \hline \end{array}$

27. $\begin{array}{r} 3\frac{1}{3}\,M - 5(a+b) \\ 2\frac{1}{2}\,M - 2(a+b) \\ \hline \end{array}$

Honor Work

28. $\begin{array}{r} a + bx \\ b - cx \\ \hline \end{array}$

29. $\begin{array}{r} cy + z \\ y - dz \\ \hline \end{array}$

30. $\begin{array}{r} ax + bx^2 \\ bx - 2\,x^2 \\ \hline \end{array}$

31. $\begin{array}{r} m(a+b) \\ n(a+b) \\ \hline \end{array}$

32. $\begin{array}{r} 2\,amn + 3\,x^a \\ 2\,bmn - 2\,x^a \\ \hline \end{array}$

33. $\begin{array}{r} a\sqrt{x+y} + abx \\ \sqrt{x+y} + acx \\ \hline \end{array}$

MULTIPLICATION OF POLYNOMIALS

In the following illustration, notice that algebraic multiplication is like arithmetic multiplication. In algebra, however, since we do not carry numbers to the next column, we usually begin at the left instead of the right.

$$\begin{array}{r} 413 \\ 21 \\ \hline 413 \\ 826 \\ \hline 8673 \end{array}$$

$$\begin{array}{r} 4\,t^2 + t + 3 \\ 2\,t + 1 \\ \hline 4\,t^2 + t + 3 \\ 8\,t^3 + 2\,t^2 + 6\,t\phantom{{}+3} \\ \hline 8\,t^3 + 6\,t^2 + 7\,t + 3 \end{array}$$

Usual Arrangement

$$\begin{array}{r} 4\,t^2 + t + 3 \\ 2\,t + 1 \\ \hline 8\,t^3 + 2\,t^2 + 6\,t\phantom{{}+3} \\ 4\,t^2 + t + 3 \\ \hline 8\,t^3 + 6\,t^2 + 7\,t + 3 \end{array}$$

Arrange the polynomial in descending (or ascending) powers of some letter. Then multiply each term of one by every term of the other as in arithmetic, and add like terms.

Check by substituting a number for each letter in the given exercise and in the answer.

Class Exercises

1. $(x+1)(x+3)$
2. $(x-2)(x-2)$
3. $(y+4)(y-2)$
4. $(x-1)(x+2)$
5. $(a+3)(a-3)$
6. $(b-5)(b-7)$

7. $(2x + 3)(3x + 1)$
8. $(3y - 5)(2y + 7)$
9. $(5n - 2)(3n - 4)$
10. $(2a - 3b)(2a + 5b)$
11. $(x - 2y)(3x - 2y)$
12. $(x^2 - 2xy)(x - 3y)$
13. $(5a^2 + 2b^2)(4a^2 + 3b^2)$
14. $(2x^3y - 3yz^2)(3x^4z^2 - xz^4)$
15. $(3x - 2y)^2$
16. $(2M + 3p)^2$
17. $(x^2 + x - 2)(x + 3)$
18. $(a^2 - 3a + 5)(a - 2)$
19. $(y^2 - y - 3)(y - 4)$
20. $(3y - y^2 + 2)(2 + y)$
21. $(3m^2 - 2n^2 + mn)(2m - 3n)$
22. $(x^2 - 2xy + 4y^2)(x + 2y)$
23. $(3x - 5 + 2x^2)(3x - 1)$
24. $(x^4 + x^2y^2 + y^4)(x^2 - y^2)$
25. $(3x^2 - 4)(2 + 3x)$
26. $(x + 2)(x - 3)(x - 1)$
27. $(x^3 - x - 4)(x^2 - 2x)$
28. $(2a + 3b + c)(2a - b)$
29. $(2x + 3y - 4)(xy - 1)$
30. $(x^4y^3 - 2x^2y^4 + 3y^5)(x^3y^3 - 2xy^4)$
31. $(2x - y)^3$
32. $(3R - 2)^3$

Optional Exercises

33. $(3a^2 + 2a - 2)(a^2 - 3a + 1)$
34. $(2x - y - 3z)(x + 2y - z)$
35. $(b^2 + bc + c^2)(b^2 - bc + c^2)$
36. $(x^4 + 2x^3 + 4x^2 + 8x + 16)(x - 2)$
37. $(a^2 - 2ab + b^2)^2$
38. $(x^2 - 2xy + y^2 - z^2)(x + y - z)$
39. $(5a - 2a^2 + 1 - a^3)(a - a^2 - 3)$
40. $(x^4y^2 - 2x^3y^3 + 3x^2y^4)(x^2 + xy - 2y^2)$
41. $(n^5 - n^3 - n + 1)(n^4 - n^3 - 2)$
42. $(2x^3 - x^2 - 3)^2$

Honor Work

43. $(x^n + 3)(x^n - 2)$
44. $(y^{a+1} - 2y)(y^a + 3)$
45. $(z^{3a} - 4)(z^{3a} + 4)$
46. $(x^{\frac{1}{2}} - 1)(x^{\frac{1}{2}} + 1)$
47. $(a^{\frac{1}{3}} - 3)(a^{\frac{1}{3}} + 2)$
48. $(m^{\frac{2}{3}} - p^{\frac{1}{5}})(m^{\frac{2}{3}} + 3p^{\frac{1}{5}})$
49. $(x^{n+2} - 2x^n - x^{n-2})(x^2 - 3)$
50. $(a^x - b^y)^2$
51. $(x^{2n} + y^{4n})(x^n + y^{2n})(x^n - y^{2n})$
52. $(b^{\frac{2}{3}} - b^{\frac{1}{3}} + 1)(b^{\frac{1}{3}} + 1)$
53. $(k^{\frac{1}{2}} - n^{\frac{1}{2}})^2$
54. $(y + y^{\frac{2}{3}} + y^{\frac{1}{3}} + 1)(y^{\frac{1}{3}} - 1)$

DIVISION OF POLYNOMIALS

Division of polynomials in algebra is practically the same as long division in arithmetic. Compare these examples:

$$\begin{array}{rl} & 21 \\ \cline{2-2} 23\big) & 483 \\ & \underline{46} \\ & 23 \\ & \underline{23} \end{array} \qquad\qquad \begin{array}{rl} & \phantom{4x^2+{}}2x+1 \\ \cline{2-2} 2x+3\big) & 4x^2+8x+3 \\ & \underline{4x^2+6x} \\ & \phantom{4x^2+{}}2x+3 \\ & \phantom{4x^2+{}}\underline{2x+3} \end{array}$$

1. Arrange the terms of both dividend and divisor in descending (or ascending) powers of some letter.

2. Divide the first term of the divisor into the first term of the dividend, and write this answer as the first term of the quotient.

3. Multiply the whole divisor by this quotient, and subtract from the dividend.

4. Repeat this process using the remainder as the dividend.

Illustration. Divide $6x^2 - 19x + 15$ by $2x - 3$.

$$\begin{array}{rll} & \phantom{6x^2-{}}3x-5 & \\ \cline{2-2} 2x-3\big) & 6x^2-19x+15 & \text{Divide } 2x \text{ into } 6x^2\text{, getting } 3x. \\ & \underline{6x^2-9x\phantom{{}+15}} & \text{Multiply } 3x \text{ by } 2x-3\text{, and subtract.} \\ & -10x+15 & \text{Divide } 2x \text{ into } -10x\text{, getting } -5. \\ & \underline{-10x+15} & \text{Multiply } -5 \text{ by } 2x-3\text{, and subtract.} \end{array}$$

Class Exercises

Divide and check by multiplication:

1. $a^2 + 9a + 20$ by $a + 4$
2. $b^2 - 7b + 10$ by $b - 2$
3. $c^2 - 5c + 6$ by $c - 3$
4. $m^2 - 3m - 10$ by $m - 5$
5. $p^2 - 5p - 14$ by $p + 2$
6. $x^2 + 9x - 22$ by $x + 11$
7. $n^2 - n - 56$ by $n - 8$
8. $y^2 + 10y - 24$ by $y + 12$
9. $z^2 - 8z + 15$ by $z - 3$
10. $k^2 + k - 12$ by $k - 3$
11. $\dfrac{4x^2 + 12x + 9}{2x + 3}$
12. $\dfrac{y^2 + z^2 - 2yz}{y - z}$

13. $\frac{4x^2-5x-6}{x-2}$

14. $\frac{12a^2-6-a}{2+3a}$

15. $\frac{5-3x-2x^2}{2x+5}$

16. $\frac{a^4-27b^2+6a^2b}{a^2-3b}$

17. $\frac{8m^2+24-38m}{4m-3}$

18. $\frac{20x^2+20y^2+41xy}{5y+4x}$

19. $\frac{10K-4+6K^2}{3K-1}$

20. $\frac{16-27x^2-6x}{2-3x}$

21. $\frac{4x^2-1}{2x-1}$

22. $\frac{25a^2-9b^2}{5a+3b}$

23. $\frac{16m^2-49p^2}{4m-7p}$

24. $\frac{a^4b^2-5a^2b-24}{a^2b-8}$

25. $\frac{16x^6y^4+14x^3y^2-15}{8x^3y^2-5}$

26. $\frac{9a^4+49b^2-42a^2b}{3a^2-7b}$

27. $\frac{6r^4s^6-6t^2-5r^2s^3t}{2r^2s^3-3t}$

28. $\frac{8x^5-10xy^4+11x^3y^2}{8x^3-5xy^2}$

29. $\frac{9x^2-16y^2}{4y-3x}$

30. $\frac{4a^3-25ab^4}{2a+5b^2}$

Optional Exercises

31. $\frac{3a^3-5a^2+a+1}{a-1}$

32. $\frac{15x^3+x^2y-3xy^2+2y^3}{3x+2y}$

33. $\frac{6n^4-5n^3-4n^2+23n-20}{2n^2-3n+4}$

34. $\frac{2a^4-a^3b-17a^2b^2+13ab^3+3b^4}{2a^2-5ab-b^2}$

35. $\frac{8m^4+14m^3-15m^2+13m-2}{2m^2+5m-1}$

36. $\frac{x^3-y^3}{x-y}$

37. $\frac{8a^3-27b^3}{2a-3b}$

38. $\frac{x^5+y^5}{x+y}$

39. $\frac{r^4+r^2s^2+s^4}{r^2-rs+s^2}$

40. $\frac{a^{10}+32b^5c^{15}}{a^2+2bc^3}$

Honor Work

Divide:

41. $\frac{x^{2a}-y^{2b}}{x^a-y^b}$

42. $\frac{t^{2a-2}-1}{t^{a-1}+1}$

43. $\dfrac{z^{2n+3} + 12z^{2n+2} + 27z^{2n+1}}{z+3}$

44. $\dfrac{x^2 + 2xy + y^2 - z^2}{x + y - z}$ 45. $\dfrac{a^3 + 3a^2b + 3ab^2 + b^3 + c^3}{a + b + c}$

46. $\dfrac{x^3 + y^3 + z^3 - 3xyz}{x + y + z}$

In these exercises find the first four terms of the quotient in ascending powers of x.

47. $\dfrac{1}{1-x}$ 48. $\dfrac{4}{1+2x}$ 49. $\dfrac{1+x^2}{1+x}$ 50. $\dfrac{4}{2+x+x^2}$

DIFFERENCE OF TWO SQUARES

Illustration. Multiply $a + b$ by $a - b$.

If we actually perform the multiplication, we get:

$$(a + b)(a - b) = a^2 - b^2$$

Rule. The product of the sum and difference of two numbers is the difference of their squares.

Exercises

Find the answers by inspection:

1. $(c + d)(c - d)$
2. $(m + n)(m - n)$
3. $(x - y)(x + y)$
4. $(x + 5)(x - 5)$
5. $(2a - b)(2a + b)$
6. $(3x + 4y)(3x - 4y)$
7. $(3t - 1)(3t + 1)$
8. $(2x^2 + 3)(2x^2 - 3)$
9. $(7 - 3y)(7 + 3y)$
10. $(a^2 + b^3)(a^2 - b^3)$
11. $(xy^2 - 4)(xy^2 + 4)$
12. $(x + .1)(x - .1)$
13. $(3a - .05)(3a + .05)$
14. $(.7x + .01)(.7x - .01)$
15. $(t^5 - 3v^2)(t^5 + 3v^2)$
16. $(s + \frac{1}{4})(s - \frac{1}{4})$

Illustration. Multiply: 47×53

$$\begin{aligned} &47 \times 53 \\ &= (50 - 3)(50 + 3) \\ &= 2500 - 9 \\ &= 2491 \end{aligned}$$

If the number half way between the two numbers to be multiplied ends in 0, this method is often much easier than multiplying out the numbers, and it has the advantage that it can more easily be done mentally.

Exercises

Find the product mentally:

1. 39×41
2. 56×64
3. 32×28
4. 37×43
5. 97×103
6. 78×82
7. 81×79
8. 104×96
9. 123×117
10. 76×84
11. 124×116
12. 85×75
13. 71×49
14. 115×105
15. 1203×1197

16. Charles Drake and Company sold this bill of goods:
48 doz. eggs at 52¢ a doz.
73 bu. potatoes at 67¢ a bu.
102 yds. cloth at 98¢ a yd.
31 lbs. coffee at 29¢ a lb.
64 lbs. tea at 56¢ a lb.

Find the total amount of the bill.

FACTORING THE DIFFERENCE OF TWO SQUARES

What is the product of $a + b$ and $a - b$? What then are the factors of $a^2 - b^2$?

Rule. When an expression is made up of two squares separated by a minus sign, take the square root of each square. One factor is the sum of these square roots, and the other factor is their difference.

$$a^2 - b^2 = (a + b)(a - b)$$

Class Exercises

Factor:

1. $x^2 - y^2$
2. $m^2 - n^2$
3. $x^2 - 9$
4. $c^2 - 4$
5. $k^2 - 1$
6. $16 - a^2$
7. $9 - x^2$
8. $4r^2 - 1$
9. $25x^2 - 9$
10. $v^2 - 4t^2$
11. $x^4 - y^2$
12. $a^6 - b^4$
13. $9a^2 - 16b^2$
14. $4x^2 - 25a^2$
15. $36p^2 - 25y^2$
16. $9a^2b^4 - 1$
17. $49 - 16s^2t^6$
18. $36x^6 - 49y^4$
19. $a^4 - 9b^2$
20. $4x^4 - y^{10}$
21. $9a^2 - 64b^4c^6$

22. $x^2 - .01$
23. $y^2 - \frac{1}{4}$
24. $z^4 - .09$
25. $.01\,x^2 - 1$
26. $4\,m^2 - .25\,y^4$
27. $\frac{1}{9} - 4\,x^2$
28. $\frac{9}{16} - m^2$

29. $\frac{x^2}{25} - \frac{1}{4}$
30. $y^6 - .0121$
31. $100\,c^4 - .01\,d^6$
32. $1.44 - x^8$
33. $d^2 - \frac{25}{49}$

34. $\frac{4}{x^2} - \frac{9}{y^2}$
35. $\frac{a^2}{x^2} - \frac{b^2}{y^2}$
36. $\frac{.01}{z^2} - \frac{49}{64}$

Optional Exercises

Find the prime factors:

37. $a^4 - b^4$
38. $x^4 - a^4$
39. $x^4 - 16$
40. $x^4 - 1$

41. $x^8 - 1$
42. $16\,x^4 - 81\,y^4$
43. $ax^2 - a$
44. $b^3 - b$

45. $2\,x^6 - 18$
46. $27\,x^3 - 12\,xy^4$
47. $x^5 - x$
48. $\pi R^2 - \pi r^2$

Honor Work

49. $(a + b)^2 - 1$
50. $(a - x)^2 - b^2$
51. $(x - y)^2 - 4$
52. $1 - (x + y)^2$
53. $4 - (a - b)^2$
54. $9 - 4(x + y)^2$

55. $(x + a)^2 - (y + b)^2$
56. $(x - 3\,y)^2 - (4 + z)^2$
57. $(2\,a - 3\,b)^2 - (3\,x - 4\,d)^2$
58. $4(a + b)^2 - 9(c + d)^2$
59. $x^{2a} - y^{2b}$
60. $x^{2n+3} - x^3$

Applied Problems

Illustration 1. Find the value of $243^2 - 242^2$.

$$\begin{aligned} &243^2 - 242^2 \\ &= (243 + 242)(243 - 242) \\ &= 485 \times 1 = 485 \end{aligned}$$

Illustration 2. Find the value of $562^2 - 438^2$.

$$\begin{aligned} &562^2 - 438^2 \\ &= (562 + 438)(562 - 438) \\ &= 1000 \times 124 \\ &= 124{,}000 \end{aligned}$$

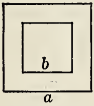
If the difference of the two numbers is small, or if either the sum or the difference ends in 0, this method is very easy and can often be done mentally. In most cases it reduces the labor of multiplying.

Do these exercises without multiplying out the squares:

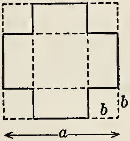
61. $36^2 - 26^2$
62. $148^2 - 52^2$
63. $435^2 - 434^2$
64. $1344^2 - 1342^2$
65. $678^2 - 322^2$
66. $8653^2 - 8649^2$
67. $3472^2 - 3272^2$
68. $3.14 \times 551^2 - 3.14 \times 549^2$
69. $3.69^2 - 2.31^2$
70. $3.14 \times 156^2 - 3.14 \times 144^2$

71. The figure shows a metal square, having a side a, from which a square, having a side b, has been cut.

(*a*) Write a formula for the area A in terms of a and b.
(*b*) Find the area of the metal if $a = 3.84$ and $b = .84$.
(*c*) Find the area if $a = 7.83$ and $b = 2.17$.



Ex. 71



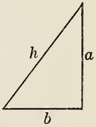
Ex. 72

72. A tin pan is made from a square of tin, side a, by cutting from each corner a square, side b, and folding along the dotted lines.

(*a*) Write a formula for the area of the outside of the pan in terms of a and b.

(*b*) Find the number of square inches of tin used to make a pan if $a = 9.38$ in. and $b = 2.19$ in.

(*c*) Find the area if $a = 73.6$ and $b = 3.2$, if $a = .83$ and $b = .065$.



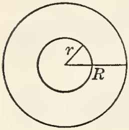
73. We have already used the fact that in a right triangle the square of a leg b equals the square of the hypotenuse h minus the square of the other leg a.

(*a*) Write a formula for finding a leg of a right triangle when the hypotenuse and the other leg are known.

(*b*) Find b for each of these triangles:

$h = 41$ and $a = 40$ $\quad$ $h = 37$ and $a = 35$
$h = 626$ and $a = 624$ $\quad$ $h = 743$ and $a = 257$

74. The area of a washer is found by subtracting the area of the small circle from that of the large circle.



(*a*) Using the letter π for 3.1416, write a formula for the area of a washer whose radii are R and r.

(*b*) Find the area of a washer if $R = 1.75$, $r = 1.25$, and $\pi = 3.14$.

How to find prime factors. 1. Look for a monomial factor. If there is one, remove it.

2. See if the remaining expression is the difference of two squares.

3. See if any of these factors can still be factored again.

4. Your answer will consist of all factors that cannot be factored again.

Illustration. Factor: $3x^5 - 48x$.

$$\begin{aligned} & 3x^5 - 48x \\ &= 3x(x^4 - 16) \\ &= 3x(x^2 + 4)(x^2 - 4) \\ &= 3x(x^2 + 4)(x + 2)(x - 2) \end{aligned}$$

Exercises

Factor:

1. $a^3 - a$
2. $t^6 - t^4$
3. $3a^2 - 3b^2$
4. $5a^2b - 20b^3$
5. $2 - 8y^2$
6. $ax^2 - ay^2$
7. $18z^2 - 8$
8. $x^3y - xy^3$
9. $45 - 5n^2$
10. $9x^3 - x$
11. $a^4 - b^4$
12. $16x^4 - 1$
13. $b^8 - 81$
14. $x^4 - 1$
15. $y^{12} - 16$
16. $k^5 - k$
17. $x^5y^2 - xy^6$
18. $z^4 - z^8$
19. $3a^4b^8 - 48$
20. $\pi R^2 - \pi r^2$
21. $x^{16} - 1$
22. $a^{16} - b^{16}$
23. $\frac{1}{2}x^3 - \frac{1}{2}x$
24. $.5a^2 - 2$
25. $2\pi R^2h - 2\pi r^2h$
26. $\frac{1}{3}x^2 - 3$
27. $7y^9 - 7y$
28. $c^2 - .01$
29. $m^4 - .0016$
30. $x^{4n+3} - x^3$

31. Using fractional exponents, factor as the difference of two squares:

(*a*) $x - 1$ (*b*) $a - 4$ (*c*) $r - s$

32. Find the prime factors of $x^n - y^n$ if:

(*a*) $n = 2$ (*b*) $n = 4$ (*c*) $n = 8$

REDUCING FRACTIONS

What effect does it have on the value of a fraction if I multiply its numerator and denominator by the same number?

Does the expression $\frac{2}{7}$ equal $\frac{6}{21}$? Does the expression $\frac{2}{3}$ equal $\frac{10}{15}$?

What effect does it have on the value of a fraction if I divide its numerator and denominator by the same number?

Does the expression $\frac{8}{12}$ equal $\frac{2}{3}$? Does the expression $\frac{5}{20}$ equal $\frac{1}{4}$?

What effect does it have on the value of a fraction if I add the same number to its numerator and denominator?

Does the expression $\frac{2}{3}$ equal $\frac{4}{5}$? Does the expression $\frac{1}{4}$ equal $\frac{3}{6}$?

What effect does it have on the value of a fraction if I subtract the same number from its numerator and denominator?

Does the expression $\frac{5}{7}$ equal $\frac{1}{3}$? Does the expression $\frac{6}{11}$ equal $\frac{5}{10}$?

Rule. We can reduce a fraction to lower terms by dividing its numerator and denominator by any of their common factors.

Cancelling means dividing by a common *factor*. We can reduce $\frac{9}{15}$ to $\frac{3}{5}$ because 3 is a *factor* of both 9 and 15. Can we reduce $\frac{21}{23}$ to $\frac{1}{3}$ by cancelling the 2's? Why not? Notice that 21 is 2 tens + 1. Can we reduce $\frac{2t + 1}{2t + 3}$ by dividing the $2t$ from the numerator and the denominator? Why not? Is $2t$ a factor of $2t + 1$? Of $2t + 3$?

Illustration 1. Reduce $\frac{9}{15}$ to lowest terms.

$$\frac{\overset{3}{\cancel{9}}}{\underset{5}{\cancel{15}}} = \frac{3}{5}$$

We can divide 3 into 9 and 15 because 3 is a factor of both 9 and 15.

Illustration 2. Reduce $\frac{3a^2 - 3x^2}{6a^2 + 6ax}$ to lowest terms.

Factor: $$\frac{3a^2 - 3x^2}{6a^2 + 6ax} = \frac{\cancel{3}\cancel{(a+x)}(a-x)}{\underset{2}{\cancel{6}}\,a\cancel{(a+x)}}$$

Cancel like factors: $$= \frac{a-x}{2a}$$

Exercises

1. Which of these fractions can be reduced and which is in its lowest terms already: $\frac{33}{43}$? $\frac{21}{35}$? $\frac{32}{52}$? $\frac{27}{47}$?

2. Tell what was done to the left member to obtain the right member:

$$\frac{8}{12} = \frac{2}{3} \qquad \frac{6v}{9w} = \frac{2v}{3w} \qquad \frac{xyz}{txz} = \frac{y}{t} \qquad \frac{x^3y^2}{x^2y^4} = \frac{x}{y^2}$$

Reduce these fractions to their lowest terms:

3. $\frac{18}{24}$

4. $\frac{ax}{ay}$

5. $\frac{x^2}{x^5}$

6. $\frac{-2x^3}{3x^2}$

7. $\frac{-5a^3}{-15a^4}$

8. $\frac{8ax^3}{-6x^2y}$

9. $\frac{-a^2b^3c}{a^3bc^5}$

10. $\frac{6x^6}{2x}$

11. $\frac{ax + ay}{bx + by}$

12. $\frac{3x^3 - 3x^2}{6x^2y - 6xy}$

13. $\frac{4n^2 - 6n}{2n^2}$

14. $\frac{12x^2y^3}{3xy - 6xy^2}$

15. $\frac{3x + 12}{2x + 8}$

16. $\frac{a - b}{a^2 - b^2}$

17. $\frac{5c^2 + 10c}{15cx + 30x}$

18. $\frac{6y + 2}{3 + 9y}$

19. $\frac{m - n}{m^2 - n^2}$

20. $\frac{a^2 - b^2}{a^2 + ab}$

21. $\frac{3y + 6}{y^2 - 4}$

22. $\frac{3x + 3y}{6x^2 - 6y^2}$

23. $\frac{x - y}{y - x}$

24. $\frac{x^3 - 4x}{4 - 2x}$

25. $\frac{c^2 - \frac{1}{4}}{c + \frac{1}{2}}$

26. $\frac{\pi R^2 - \pi r^2}{2\pi R - 2\pi r}$

How to find the product of two fractions. We multiply the numerators of two fractions together for the numerator of the product, and multiply the denominators together for the denominator of the product. We can think of an integer as a numerator with 1 as its denominator.

Illustrations.

$$3\left(\frac{2}{7}\right)=\frac{6}{7} \qquad c\left(\frac{a}{b}\right)=\frac{ca}{b} \qquad \frac{2}{\cancel{3}}\cdot\frac{\cancel{3}}{5}=\frac{2}{5} \qquad \frac{2}{\cancel{a}}\cdot\frac{\cancel{a}}{b}=\frac{2}{b} \qquad \frac{ax}{by}\cdot\frac{y^2}{x^2}=\frac{ay}{bx}$$

$$\frac{6x-12}{2x+4}\cdot\frac{x^2-4}{9x-18}=\frac{\overset{\cancel{3}}{\cancel{6}}(x-2)\cancel{(x+2)}\cancel{(x-2)}}{\cancel{2}\cancel{(x+2)}\underset{3}{\cancel{9}}\cancel{(x-2)}}=\frac{x-2}{3}$$

To multiply fractions:

1. Find the prime factors of all numerators and denominators.

2. Write as a single fraction with all the numerators above the line and all the denominators below the line.

3. Divide numerator and denominator by common factors.

Never cancel anything except a factor of the whole expression. A term or a factor of a term must never be cancelled.

Exercises

Multiply, and reduce to lowest terms:

1. $3\cdot\frac{4}{15}$

2. $\frac{5}{6}\cdot\frac{9}{10}$

3. $3(-\frac{1}{2})$

4. $a\left(\frac{b}{a^2}\right)$

5. $xy\cdot\frac{x}{y}$

6. $\frac{3a}{4b}\cdot\frac{2b}{9a}$

7. $\frac{ax}{by^2}\cdot\frac{aby}{x}$

8. $\frac{5x}{y}\cdot\frac{-2y}{x}$

9. $\frac{-3}{x^2y^3}\cdot\frac{2x^3y}{-5}$

10. $\frac{5x^2}{4yz}\cdot\frac{8yz^2}{15x^2}$

11. $\frac{axy}{bxz}\cdot\frac{byz}{ay^2}$

12. $\frac{x}{y}\cdot\frac{y}{z}\cdot\frac{z}{x}$

13. $\frac{a^2}{b^2}\cdot\frac{bc}{ac}\cdot\frac{ab}{a^2}$

14. $\frac{3xz}{4y^2}\cdot\frac{2yz}{3x^2}\cdot\frac{2xy}{z^2}$

15. $2\cdot\frac{-x}{3}\cdot\frac{5}{x}$

16. $\frac{x^5}{y^4}\cdot\frac{y^3}{z^2}\cdot\frac{z}{x^3}$

17. $\frac{3a^3b^4}{2c} \cdot 4c \cdot \frac{1}{6ab^3}$

18. $\frac{2x^2}{x+1} \cdot \frac{x^2-1}{6x}$

19. $3m \cdot \frac{5}{m^2n} \cdot 2n^2$

20. $\frac{ax-ay}{ab} \cdot \frac{bx+by}{x^2-y^2}$

21. $(x-y) \cdot \frac{3x}{x^2-y^2}$

22. $\frac{2x-2y}{3x+3y} \cdot \frac{9x^2+9xy}{4x^2-4yx}$

23. $\frac{(a+b)^2}{a^2-9b^2} \cdot \frac{(a-3b)^2}{a^2-b^2}$

24. $\frac{2x^2-x}{4x^2-1} \cdot \frac{4x+2}{6x}$

25. $\frac{3x^2y-12y^3}{5xy^2+10y^3} \cdot \frac{2x^3y^2}{6x^3-12x^2y}$

26. $\frac{m+n}{m-n} \cdot \frac{m(m-n)^2}{2m^3-2mn^2}$

27. $\frac{mx+my}{px-py} \cdot \frac{nx-ny}{rx+ry}$

28. $5a \cdot \frac{9a^2-b^2}{18a^2b+6ab^2} \cdot 3b$

29. $\frac{a^2-16b^2}{(a-2b)^2} \cdot \frac{3a^3b-6a^2b^2}{a^2+4ba}$

30. $\frac{z^2-4}{9+3z} \cdot \frac{z^2-9}{4z+2z^2}$

DIVISION OF FRACTIONS

In algebra the division of fractions is performed in the same way as in arithmetic, that is, we invert the divisor fraction and proceed as in multiplication.

Illustrations.

$$\frac{3}{8} \div \frac{9}{4} = \frac{3}{8} \cdot \frac{4}{9} = \frac{1}{6}$$

$$\frac{a}{b^2} \div \frac{2a^3}{b} = \frac{a}{b^2} \cdot \frac{b}{2a^3} = \frac{1}{2a^2b}$$

$$\frac{3x-6}{4x+6} \div \frac{4x-8}{2x^2+3x} = \frac{3(x-2)}{2(2x+3)} \cdot \frac{x(2x+3)}{4(x-2)} = \frac{3x}{8}$$

Exercises

Divide:

1. $\frac{1}{x} \div \frac{1}{y}$

2. $\frac{1}{x} \div \frac{1}{x}$

3. $\frac{x^3}{y^3} \div \frac{x}{y}$

4. $\frac{-a}{bc} \div \frac{a^2}{c^2}$

5. $\frac{xy}{z^3} \div \frac{x^2y^3}{-z}$

6. $5m \div \frac{m^2}{2}$

7. $\frac{c^2d^4}{10t} \div \frac{c^2d^4}{15t}$

8. $\frac{xy}{3} \div (-3x)$

9. $\frac{15xy^2z^3}{14m^2n} \div \frac{10x^2yz^3}{21mn^3}$

10. $\frac{a^2+ab}{a-b} \div \frac{a^2-ab}{a-b}$

11. $\frac{a-1}{a-3} \div \frac{3a-3}{2a-6}$

12. $\frac{x^4-1}{x-1} \div \frac{x^2+1}{x+1}$

13. $\frac{xy-y^2}{x} \div \frac{x^2-xy}{y}$

14. $\frac{4a+4b}{3a-6b} \div \frac{8a+8b}{9a-18b}$

15. $\frac{m^2-4}{m^2+2m} \div \frac{mn-2n}{3mn}$

16. $\frac{n-3}{n+3} \div \frac{3-n}{3+n}$

17. $\frac{16s^2-1}{s+4s^2} \div (2s^2-8s^3)$

ADDITION OF FRACTIONS

Add these fractions:

1. $\frac{1}{8}+\frac{3}{8}$

2. $\frac{2}{7}+\frac{3}{7}$

3. $\frac{3}{5}-\frac{4}{5}$

4. $\frac{2}{x}+\frac{3}{x}$

5. $\frac{4}{y}-\frac{2}{y}$

6. $\frac{5}{z}-\frac{1}{z}$

7. $\frac{a}{x}+\frac{b}{x}$

8. $\frac{2c}{y}-\frac{d}{y}$

9. $\frac{5m}{z}-\frac{2m}{z}$

Notice that if the denominators of fractions are the same, we add the fractions by adding their numerators and writing this result over this common denominator.

Find the missing numerators:

10. $\frac{2}{3}=\frac{}{6}$

11. $\frac{4}{5}=\frac{}{15}$

12. $\frac{1}{2}=\frac{}{8}$

13. $\frac{4}{x}=\frac{}{3x}$

14. $\frac{3}{y}=\frac{}{ay}$

15. $\frac{1}{a}=\frac{}{ab}$

16. $\frac{a}{b}=\frac{}{2b}$

17. $\frac{c}{d}=\frac{}{dx}$

18. $\frac{x}{y}=\frac{}{y^2}$

Notice that we can multiply the denominator of a fraction by any number we please without changing the value of the fraction provided we multiply the numerator by that same number.

Perform the additions:

19. $\frac{2}{3}+\frac{1}{5}=\frac{}{15}+\frac{}{15}=\frac{}{15}$

20. $\frac{3}{x}+\frac{2}{y}=\frac{}{xy}+\frac{}{xy}=\frac{}{xy}$

21. $\frac{a}{x} + \frac{b}{y} = \frac{\quad}{xy} + \frac{\quad}{xy} = \frac{\quad}{xy}$

22. $\frac{a}{b} + \frac{c}{d}$ 23. $\frac{2a}{z} - \frac{a}{w}$ 24. $\frac{3n}{x} - \frac{2p}{3y}$

Notice that to add fractions whose denominators are not alike:

1. We find the smallest number into which each denominator will go a whole number of times. This is called the ***lowest common denominator*** or L.C.D.
2. We multiply each denominator by that factor that will give the L.C.D.
3. We multiply each numerator by the same number by which its denominator was multiplied.
4. We add the numerators and write the result over the L.C.D.

Exercises

Add:

1. $\frac{x}{2} + \frac{x}{3}$
2. $\frac{4y}{2} - \frac{3y}{2}$
3. $\frac{a}{3} - \frac{2a}{5}$
4. $\frac{x}{y} - \frac{z}{v}$
5. $\frac{m}{x} + \frac{n}{y}$
6. $\frac{2b}{3x} + \frac{3b}{2y}$
7. $\frac{x}{y^2} - \frac{y}{x^2}$
8. $3 + \frac{2}{5}$
9. $a + \frac{b}{c}$
10. $\frac{a}{x} + \frac{3+2a}{x}$
11. $\frac{b}{y} - \frac{2b+1}{y}$
12. $\frac{c}{z} - \frac{c-d}{z}$
13. $\frac{m-n}{3} + \frac{n-m}{3}$
14. $\frac{c-2}{5} - \frac{3-2c}{5}$
15. $\frac{x}{3} - \frac{2x-1}{6}$
16. $\frac{3w-2}{6} + \frac{2w-1}{8}$
17. $1 + \frac{2b+1}{3}$
18. $\frac{x}{ab} - \frac{ax}{b^2c} + \frac{bx}{ac^2}$
19. $a - 2 - \frac{1-2a}{5}$

20. $\frac{x}{3} - 2x + \frac{3x+2}{6}$

21. $\frac{x+3}{2} + \frac{2x-1}{3} - \frac{4-3x}{6}$

22. $\frac{2a-3}{6} + \frac{3a-2}{4} - a$

23. $\frac{x-y}{xy} + \frac{y-z}{yz} + \frac{z-x}{zx}$

Illustration. Add: $\frac{x-2}{x^2-9} + \frac{3}{x^2+3x}$

$$\begin{aligned}
&\frac{x-2}{x^2-9} + \frac{3}{x^2+3x} \\
&= \frac{x-2}{(x+3)(x-3)} + \frac{3}{x(x+3)} \\
&= \frac{x(x-2)}{x(x+3)(x-3)} + \frac{3(x-3)}{x(x+3)(x-3)} \\
&= \frac{x^2-2x+3x-9}{x(x+3)(x-3)} \\
&= \frac{x^2+x-9}{x(x+3)(x-3)}
\end{aligned}$$

24. $\frac{x^2-1}{x^2-2x} + \frac{x+3}{x}$

25. $\frac{4}{x+1} + \frac{3}{x-1}$

26. $\frac{5}{y^2-9} + \frac{2}{y^2+3y}$

27. $\frac{x-2}{x-3} - \frac{x^2-2x}{x^2-9}$

28. $\frac{3}{4r^2-9} - \frac{4}{6r+9}$

29. $\frac{5z-3}{6z+9} - \frac{2+3z}{4z+6}$

30. $\frac{5}{3n+12} + \frac{3}{6n-3}$

31. $\frac{r^2+2s^2}{r^2-s^2} - \frac{r-s}{r+s}$

REVIEW OF FRACTIONAL EQUATIONS

Class Exercises

Solve and check:

1. $\frac{x}{2} = 5$

2. $\frac{x}{3} = -2$

3. $\frac{y}{-5} = 2$

4. $\frac{2z}{7} = 6$

5. $\frac{3a}{5} = \frac{4}{7}$

6. $\frac{2}{b} = 1$

7. $\frac{6}{3x} = 2$

8. $\frac{x}{3} = \frac{3}{4}$

9. $2 = \frac{3}{x}$

10. $\frac{3}{5} = \frac{2}{z}$

11. $\frac{x}{3} + \frac{x}{2} = 5$

12. $\frac{y}{4} - \frac{y}{2} = 3$

13. $\frac{k}{3} - \frac{k}{6} = -1$

14. $\frac{a}{3} + 4 = 6$

15. $\frac{m}{5} - 3 = 2$

16. $\frac{t}{7} - 5 = 0$

17. $\frac{x}{5} - 3 = \frac{x}{8}$

18. $\frac{5y}{18} - \frac{2y}{9} = \frac{1}{3}$

19. $\frac{3}{4} = z - \frac{1}{2}$

20. $7 - \frac{r}{2} = \frac{9}{2}$

Optional Exercises

21. $\frac{x+4}{3} = 5$

22. $\frac{y-1}{2} = 3$

23. $\frac{5s-7}{6} = \frac{1}{2}$

24. $\frac{3}{x} - \frac{2}{x} = 1$

25. $4 - \frac{3+x}{2} = 0$

26. $2 - \frac{5-x}{3} = 1$

27. $\frac{5-x}{4-x} = 2$

28. $\frac{v+1}{7-v} = 1$

29. $\frac{F}{2} - \frac{F-4}{6} = 4$

30. $\frac{c+3}{3} - 1 = 2$

31. $\frac{2t+7}{5} - \frac{t-1}{6} = \frac{5}{2}$

32. $\frac{m+5}{3} + 4 = \frac{3m+2}{2}$

33. $\frac{2s-3}{4} - \frac{s+1}{6} = \frac{3}{4}$

34. $\frac{3}{5}(y+4) - \frac{1}{6}(y-2) = 1$

35. $\frac{2}{3}(y+7) - \frac{1}{5}(y-2) = 6$

36. $\frac{m+2}{2} - \frac{2m-1}{5} = 2$

37. $\frac{4x+3}{3} - 2x + 4 = \frac{16-x}{2}$

38. $\frac{5y-5}{4} + 8 - \frac{y+3}{2} = 0$

39. $\frac{7}{x-1} = \frac{5}{x-3}$

40. $\frac{4}{2y-5} = \frac{3}{y-4}$

Honor Work

41. $\frac{2x-1}{3x+2} = \frac{4x-5}{6x-2}$

42. $\frac{c-2}{c-3} = \frac{c+4}{c+9}$

43. $\frac{3t}{2t-7} = \frac{3t+21}{2t}$

44. $\frac{v-2}{v^2-4} + \frac{9}{v+2} = \frac{2}{v-2}$

45. $\frac{3m+4}{m^2+5m} - \frac{1}{m} = \frac{3}{2m+10}$

46. $\frac{3}{x-1} - 2 = \frac{3x-7}{1-x}$

47. $\frac{2}{y-2} - \frac{5}{y+2} = \frac{2}{y^2-4}$

48. $\frac{1}{z+1} - \frac{4}{z^2-1} = \frac{1}{1-z}$

FRACTIONAL EQUATIONS IN TWO UNKNOWNS

Solve:

$$\frac{3}{x} + \frac{2}{y} = 1 \qquad (1)$$

$$\frac{8}{x} - \frac{3}{y} = \frac{7}{12} \qquad (2)$$

If we clear the first equation of fractions, we shall have the equation $3y + 2x = xy$, in which all three terms have unknowns. To avoid this, it is better to get rid of a letter before clearing of fractions.

To get rid of y:

Multiply (1) by 3: $$\frac{9}{x} + \frac{6}{y} = 3$$

Multiply (2) by 2: $$\frac{16}{x} - \frac{6}{y} = \frac{7}{6}$$

Add: $$\frac{25}{x} = \frac{25}{6}$$

Now clear of fractions: $$25x = 150$$ $$x = 6$$

Substituting $x = 6$ in (1): $$\frac{3}{6} + \frac{2}{y} = 1$$ $$3y + 12 = 6y$$ $$y = 4$$

Check by substituting in both original equations:

$$\begin{array}{c|c} \frac{3}{6}+\frac{2}{4} & 1 \\ \frac{1}{2}+\frac{1}{2} & 1 \\ \multicolumn{2}{c}{1 = 1} \end{array} \qquad\qquad \begin{array}{c|c} \frac{8}{6}-\frac{3}{4} & \frac{7}{12} \\ \frac{16-9}{12} & \frac{7}{12} \\ \multicolumn{2}{c}{\frac{7}{12} = \frac{7}{12}} \end{array}$$

Exercises

Solve and check:

1. $\frac{1}{x} + \frac{1}{y} = 3$
   $\frac{1}{x} - \frac{1}{y} = 1$

2. $\frac{5}{x} - \frac{2}{y} = 3$
   $\frac{3}{x} - \frac{2}{y} = 1$

3. $\frac{3}{x} + \frac{6}{y} = \frac{7}{2}$
   $\frac{1}{x} + \frac{1}{y} = \frac{5}{6}$

4. $\frac{12}{x} + \frac{7}{y} = -4$
   $\frac{4}{x} + \frac{2}{y} = -1$

5. $\frac{4}{x} + \frac{3}{y} = 18$
   $\frac{5}{x} - \frac{4}{y} = 7$

6. $\frac{2}{x} + \frac{3}{y} = \frac{7}{4}$
   $\frac{3}{x} + \frac{2}{y} = 2$

7. $\frac{3}{2x} + \frac{2}{3y} = 17$
   $\frac{4}{3x} - \frac{1}{2y} = 2$

8. $\frac{7}{3x} - \frac{3}{2y} = 16$
   $\frac{5}{2x} - \frac{2}{3y} = 11\frac{1}{2}$

Work Problems

1. Three houses are built alike. Mr. Walsh painted his house in 8 days. It took Mr. Peach 10 days to paint his. If both should work together, how long would it take them to paint the third house?

	Time	*Part per Day*	*Part in x Days*
Mr. Walsh......	8	$\frac{1}{8}$	$\frac{x}{8}$
Mr. Peach......	10	$\frac{1}{10}$	$\frac{x}{10}$

$$\frac{x}{8} + \frac{x}{10} = 1$$

Notice that the whole piece of work is represented by 1.

2. One tractor plowed a certain number of acres in 6 hrs. Another tractor plowed an equal-sized field in 9 hrs. How long

Courtesy of J. I. Case.

PLOWING WITH A TRACTOR

In modern farming the tractor is used because with it the farmer can plow several times as fast as he could with a team of horses.

should it take both tractors together to plow a third field of the same size as each of the first 2?

3. Two machines do a piece of work in 5 hrs. Before the new machine was installed, it used to take the old machine 15 hrs. How long would it take the new machine alone?

4. The stream of water flowing into the new reservoir filled it in 50 days. Then the aqueduct to the city was opened. If this aqueduct can carry off a reservoir full of water in 45 days, how long will it be before the reservoir is empty?

5. Sarah can type a report in 6 hrs. Elizabeth takes 8 hrs. for the same work. After Sarah has worked for 2 hrs., Elizabeth is sent to help her. In how many more hours should the two girls complete the work?

6. Mr. Richmond worked 4 days on a piece of work. Then Mr. Hunt took it over and finished it in 3 days. Later he took over a similar piece of work on which Mr. Richmond had worked 6 days and finished it in $1\frac{1}{2}$ days. How long would each alone have taken to do one of those pieces of work?

	No. of Days	*Part Done by Each*	
		1st Job	*2nd Job*
Mr. Richmond....	x	$\frac{4}{x}$	$\frac{6}{x}$
Mr. Hunt........	y	$\frac{3}{y}$	$\frac{3}{2y}$

$$\frac{4}{x}+\frac{3}{y}=1$$
$$\frac{6}{x}+\frac{3}{2y}=1$$

7. Working together, Chester and Tom hoed an acre of potatoes in 3 days. On a second acre, Tom worked alone for 4 days, and then Chester, working alone, finished the work in $2\frac{1}{4}$ days. How long would each boy alone take to hoe an acre of potatoes?

8. One week John and Robert delivered their magazines in 4 hrs. The next week Robert worked alone for 3 hrs., and then John joined him, and they finished the work in 3 hrs. How long would it take each boy alone to deliver all the magazines?

Chapter 16

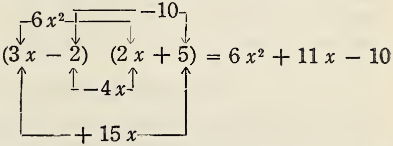
THE QUADRATIC EQUATION

A short cut in multiplication.

Multiply $3x - 2$ by $2x + 5$.

$$\begin{array}{r}3x - 2\\ 2x + 5\\ \hline 6x^2 - 4x\phantom{{}-10}\\ 15x - 10\\ \hline 6x^2 + 11x - 10\end{array}$$

From what did you get the $6x^2$? The -10? What two quantities were added to give $11x$? Where did the $-4x$ come from? The $+15x$?



1. The first term of the answer is the product of the first terms.

2. The second term of the answer is the sum of the outer and inner products.

3. The last term of the answer is the product of the last terms.

Class Exercises

Multiply:

1. $(x + 2)(x + 3)$
2. $(x + 4)(x + 6)$
3. $(x + 1)(x + 8)$
4. $(x + 5)(x + 5)$
5. $(3 + x)(4 + x)$
6. $(x - 3)(x - 1)$
7. $(x - 4)(x - 4)$
8. $(x - 9)(x - 2)$
9. $(1 - x)(2 - x)$
10. $(5 - x)(3 - x)$
11. $(x + 4)(x - 3)$
12. $(x + 3)(x - 5)$
13. $(x - 7)(x + 2)$
14. $(x + 2)(x - 2)$
15. $(6 - x)(3 + x)$

Thought Question

Can you discover an easier rule for multiplying binomials when the coefficient of x in both of them is 1?

Optional Exercises

16. $(2x + 3)(x + 5)$
17. $(3A + 2)(5A + 3)$
18. $(2b + 3)(2b + 3)$
19. $(3n - 1)(n - 5)$
20. $(5a - 3)(4a + 7)$
21. $(5m + 4)(5m - 2)$
22. $(7 - k)(4 + 3k)$
23. $(w_1 + 5)(2w_1 - 3)$
24. $(4 - n)(3 - 4n)$
25. $(5h - 4)(4h + 5)$
26. $(3ab - c)(2ab + 3c)$
27. $(4x^2 - 2y)(3x^2 + 4y)$
28. $(M^2 - pq)(3M^2 - 5pq)$
29. $(abc - 1)(3abc + 4)$
30. $(2x_1 + x_2)(3x_1 - 2x_2)$
31. $(a + 1.2)(.5a + .4)$

Honor Work

32. $(2x + \frac{1}{2})(\frac{1}{2}x - 2)$
33. $(\frac{2}{3} - 5m)(\frac{1}{5} + 3m)$
34. $(5t - 4)(.2t - .05)$
35. $(.5w + 4)(2w - .3)$
36. $(ax + 1)(a^2x - a)$
37. $(5x^n - 4)(3x^n + 5)$
38. $(3A^x - 2B^y)(2A^x - 3B^y)$
39. $(ax + b)(cx + d)$
40. $(ax + b)(cx - d)$
41. $(dw_1 + sw_2)(sw_1 + dw_2)$

FACTORING TRINOMIALS

Can you discover what two binomials I multiplied to get $x^2 + 5x + 4$? What must I have multiplied to get x^2? What could I have multiplied to get $+ 4$? Would 2 and 2 give the middle term? Would 4 and 1 do it? If the product is $+ 4$, are the signs in the factors alike or unlike? If the product had been negative, what could you say of the signs in the factors? If they are alike, must they be + or − to give $+ 5x$?

Class Exercises

Factor:

1. $x^2 + 3x + 2$
2. $x^2 + 5x + 6$
3. $x^2 + 4x + 3$
4. $x^2 - 5x + 6$
5. $x^2 - 4x + 4$
6. $x^2 - 5x + 4$
7. $x^2 - 7x + 12$
8. $x^2 - 9x + 20$
9. $x^2 + 7x + 6$
10. $x^2 - 9x + 8$
11. $x^2 - 3x - 4$
12. $x^2 + x - 2$
13. $m^2 - 5m - 6$
14. $k^2 + 3k - 28$
15. $w^2 - 4w + 3$
16. $y^2 - 5y - 14$

17. $x^2 + 8x + 7$
18. $a^2 + a - 12$
19. $n^2 - 2n - 15$
20. $R^2 - 3R - 10$

Optional Exercises

21. $2x^2 + 5x + 2$
22. $2a^2 + 5a + 3$
23. $2b^2 + 11b + 15$
24. $6M^2 + 13M + 6$
25. $2y^2 + 11y + 5$
26. $5r^2 + 17r + 6$
27. $4x^2 - 12x + 9$
28. $5x^2 - 11x + 2$
29. $15k^2 - 19k + 6$
30. $8t^2 - 22t + 15$
31. $8a^2 - 14a - 15$
32. $6m^2 - 11m - 10$
33. $12y^2 + y - 6$
34. $16x^2 + 8x - 15$
35. $6r^2 - 13r - 15$
36. $15x^2 + 19xy - 10y^2$
37. $8z^2 - 6zy - 9y^2$
38. $6b^2 + by - 35y^2$
39. $16h^2 - 34hk - 15k^2$
40. $6 - x - 12x^2$

Honor Work

Find the prime factors of:

41. $6x^2 - 15x + 6$
42. $4R^2 + 22R + 10$
43. $3x^3 + 9x^2 - 120x$
44. $18x^4 - 3x^2 - 45$
45. $x^4 + 2x^2 - 3$
46. $4y^4 - 25y^2 + 36$
47. $32m^4 + 14m^2 - 1$
48. $3x^5 - 39x^3 + 108x$
49. $x^2 - x - .24$
50. $.03x^2 - .4x - \frac{5}{3}$

THE QUADRATIC EQUATION

An equation in which the highest power of the unknown letter is its square is called a *quadratic equation*. $3x^2 + 2x + 5 = 0$ is a quadratic equation. If it is written equal to any other quantity than 0, we can always transpose so as to have the right member 0.

A *root* of an equation is the quantity that when substituted for the letter will make the equation an identity. When the equation is in the form $3x^2 + 2x + 5 = 0$, the root or answer should make the left member 0.

If any quantity is multiplied by 0, the product is 0. Consequently, if a quadratic equation can be factored, any number that makes either factor 0, will make the left member 0, and will satisfy the equation.

Illustration. Solve: $2x^2 - 3x - 2 = 0$
Factoring: $(2x + 1)(x - 2) = 0$

Since a number that makes either factor 0 will make the product 0, we can get the roots by solving the two equations:

$$\begin{array}{rclcrcl} 2x + 1 & = & 0 & \quad\text{and}\quad & x - 2 & = & 0 \\ 2x & = & -1 & & x & = & 2 \\ x & = & -\frac{1}{2} & & & & \end{array}$$

Therefore, we have two answers: $x = -\frac{1}{2}$ or 2

Exercises

Solve by factoring:

1. $x^2 - 3x + 2 = 0$
2. $x^2 - 5x + 6 = 0$
3. $x^2 - 4x + 3 = 0$
4. $x^2 - x - 2 = 0$
5. $x^2 + 3x + 2 = 0$
6. $x^2 - 5x - 6 = 0$
7. $x^2 - 12 = x$
8. $7x + x^2 = -10$
9. $3 = 2x + x^2$
10. $5x - x^2 + 6 = 0$
11. $2x^2 - 3x + 1 = 0$
12. $3x^2 - 7x + 2 = 0$
13. $6x^2 - 5x + 1 = 0$
14. $3x^2 + 8x = 3$
15. $2 - 9x - 5x^2 = 0$
16. $15x^2 + 17x + 4 = 0$
17. $6x^2 = 7x + 3$
18. $4x^2 + 7x + 3 = 0$
19. $19x - 6 = 15x^2$
20. $7x = 6x^2$

The plus or minus sign. How is it that the same equation can have two different answers? Consider $x^2 = 9$. What is the value of x? Evidently $x = 3$, for $3^2 = 9$. But x could also equal -3 for $(-3)^2$ also equals 9. So $x = +3$ or -3. Instead of writing the two answers separately, we often write $x = \pm 3$, which we understand to mean that x can equal either $+3$ or -3. We read it "x equals plus or minus three."

THE PERFECT TRINOMIAL SQUARE

By the method you have just learned or by actual multiplication, find the square of $a + b$.

$$(a + b)^2 = a^2 + 2ab + b^2$$

In the product:
The first term is the square of a.
The last term is the square of b.
The middle term is double the product of a and b.

Exercises

Square these binomials:

1. $x + y$
2. $2m + n$
3. $a - b$
4. $5x - 2y$
5. $3a - 5b$
6. $4x + 3y$
7. $6m^2 - 3y^2$
8. $5a^2b^5 + 1$

Find the square root of:

9. $4x^2 + 4x + 1$
10. $m^2 + 6m + 9$
11. $16t^2 - 8t + 1$
12. $9v^2 - 12v + 4$
13. $4y^2 + 20yz + 25z^2$
14. $1 - 8m + 16m^2$
15. $9x^2 - 30xy^2 + 25y^4$
16. $36x^2 + 60x + 25$

In a perfect square, the middle term is double the product of the square roots of the end terms.

Find the middle term that will complete the square:

17. $x^2 + \qquad + y^2$
18. $a^2 - \qquad + 4b^2$
19. $9m^2 + \qquad + 4n^2$
20. $4y^2 - \qquad + 25$
21. $9x^2 - \qquad + 49$
22. $4h^2 + \qquad + 9k^2$
23. $16y^2 + \qquad + 25z^2$
24. $64 - \qquad + 9r^2$

Illustration. Complete the square: $9x^2 - 12x +$

The first term of the square root is $3x$. Since $-12x$ is double the product of $3x$ and another quantity, we can find the other quantity by dividing $-12x$ by double $3x$ or by $6x$. The second term, then is -2, and its square is $+4$. So the square is $9x^2 - 12x + 4$.

Complete the square:

25. $x^2 + 6x +$
26. $m^2 - 10m +$
27. $k^2 - 14k +$
28. $y^2 + 8y +$
29. $4x^2 + 12x +$
30. $9x^2 - 30x +$
31. $4x^2 - 4x +$
32. $4x^2 + 2x +$

Solving the quadratic equation by completing the square. Not all quadratic equations can be solved by factoring. Consequently we must learn a method that can be used in all cases. This method is called completing the square.

Illustration 1. Solve: $x^2 - 6x - 7 = 0$

Transpose the − 7: $x^2 - 6x = 7$

Add 9 to complete the square: $x^2 - 6x + 9 = 16$

Since it is now a perfect square, take the square root of both members: $x - 3 = \pm 4$

Transpose the − 3: $x = \pm 4 + 3$

Using first + 4, then − 4: $x = 7$ or -1

Illustration 2. Solve: $5x^2 - 3x - 4 = 0$

Transpose the − 4: $5x^2 - 3x = 4$

Divide by 5: $x^2 - \frac{3}{5}x = \frac{4}{5}$

$\frac{1}{2}(\frac{3}{5}) = \frac{3}{10}$. Square and add: $x^2 - \frac{3}{5}x + \frac{9}{100} = \frac{4}{5} + \frac{9}{100}$

Combine terms: $= \frac{89}{100}$

Take the square root: $x - \frac{3}{10} = \pm \frac{1}{10}\sqrt{89}$

$$x = \pm \frac{1}{10}\sqrt{89} + \frac{3}{10}$$

$$= \frac{3 \pm \sqrt{89}}{10}$$

Exercises

Solve by completing the square:

1. $x^2 - 6x + 5 = 0$
2. $x^2 - 8x + 15 = 0$
3. $x^2 - 10x - 24 = 0$
4. $x^2 + 4x - 5 = 0$
5. $x^2 - 9x + 20 = 0$
6. $x^2 - x - 12 = 0$
7. $2x^2 - 5x - 3 = 0$
8. $3x^2 - 7x + 2 = 0$
9. $3x^2 + 8x = 3$
10. $2x^2 + 5 = 11x$
11. $x^2 - 4x - 6 = 0$
12. $x^2 + 3x - 5 = 0$
13. $x^2 - 4x = 2$
14. $x^2 + 7 = 9x$
15. $2x^2 - 6x + 3 = 0$
16. $3x^2 - 5x - 7 = 0$
17. $2x^2 + 7x = 11$
18. $5x^2 = 3x + 1$
19. $3x^2 - 11 = 4x$
20. $2x - 3x^2 = -4$
21. $5x - 2 = 2x^2$
22. $4x^2 + 1 = 4x$
23. $x^2 - 6x = 0$
24. $x^2 - \frac{15}{4}x - 1 = 0$

THE QUADRATIC FORMULA

Solve the equation: $ax^2 + bx + c = 0$, for x

Dividing by a: $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

Transposing $\frac{c}{a}$: $x^2 + \frac{b}{a}x = -\frac{c}{a}$

Completing the square: $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$

$$= \frac{b^2 - 4ac}{4a^2}$$

Taking the square root: $x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$

Transposing $\frac{b}{2a}$: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

This is a general solution, for a, b, and c can stand for any numbers. For example, if $a = 2$, $b = -5$, and $c = 3$, the equation

$ax^2 + bx + c = 0$ becomes $2x^2 - 5x + 3 = 0$, and the answer

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ becomes } x = \frac{+5 \pm \sqrt{25 - 24}}{4}$$

$$= \frac{+5 \pm 1}{4}$$

$$= \frac{3}{2} \text{ or } 1$$

Check by substituting the answers in the original equation.

HISTORICAL NOTE ON SOLVING THE QUADRATIC EQUATION: Although the quadratic equation was not entirely unknown in early times, and a few special cases were solved geometrically by Euclid (300 B.C.), the Hindu writers deserve most of the credit for the general solution. Brahmagupta (628) gave a rule equivalent to a formula, and Sridhara (1025) explained the method of completing the square. Long afterwards Harriot (1631) solved quadratic equations by factoring. Vieta (about 1580) made the advance of solving these equations analytically instead of geometrically.

The formula. We can then solve the equation:

$$ax^2 + bx + c = 0$$

where a, b, and c are any numbers by substituting in the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Illustration. Solve: $3x^2 - 4x - 4 = 0$

Here $a = 3$, $b = -4$, and $c = -4$. Then $-4ac = +48$.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{+4 \pm \sqrt{16 + 48}}{6}\text{*} \\ &= \frac{+4 \pm \sqrt{64}}{6} \\ &= \frac{+4 \pm 8}{6} = 2 \text{ or } -\frac{2}{3} \end{aligned}$$

Class Exercises

Solve by means of the formula, and check:

1. $x^2 + 3x + 2 = 0$
2. $x^2 + x - 6 = 0$
3. $x^2 - 4x + 4 = 0$
4. $2x^2 - 5x + 2 = 0$
5. $6x^2 - 2x - 4 = 0$
6. $3y^2 - 4 = 4y$
7. $4k^2 - 12k + 9 = 0$
8. $3w^2 + 2w = 8$
9. $2h^2 = 5 + 9h$
10. $17x = 12x^2 + 6$

---

* NOTE: If you always make the coefficient of x^2 positive, you can check your signs at this point by noting that the signs of $-b$ and of $-4ac$ are always the opposites of those of the second and third coefficients of the equation.

$$\begin{tikzcd}[column sep=small]
3x^2 \; - \arrow[d, leftrightarrow] & 4x & -\,4 = 0 \arrow[dll, leftrightarrow] \\
x = \dfrac{+4 \pm \sqrt{16 + 48}}{6} & &
\end{tikzcd}$$

Optional Exercises

11. $x^2 - 2x - 1 = 0$

12. $x^2 + x - 5 = 0$

13. $x^2 - 6x + 2 = 0$

14. $3x^2 - 5x - 4 = 0$

15. $2x^2 + 7x + 2 = 0$

16. $x(x - 4) = 7$

17. $\frac{x}{3} = \frac{2}{x + 1}$

18. $\frac{x}{x - 2} - \frac{1}{x} = 3$

19. $x + 7 + \frac{3}{x} = 0$

20. $\frac{x + 2}{3} = \frac{1}{x - 3}$

Honor Work

Illustration. Find the roots of $2x^2 - 4x + 1 = 0$ to the nearest hundredth.

$$\begin{aligned} x &= \frac{+4 \pm \sqrt{16 - 8}}{4} \\ &= \frac{+4 \pm \sqrt{8}}{4} \\ &= \frac{+4 \pm 2.828}{4} \\ &= \frac{6.828}{4} \text{ or } \frac{1.172}{4} \\ &= 1.707 \text{ or } .293 \\ &= 1.71 \text{ or } .29 \end{aligned}$$

$$\begin{array}{l|r} & 2.\;8\;2\;8 \\ & \sqrt{8.000000} \\ & 4 \\ \hline 48 & 4\,00 \\ & 3\,84 \\ \hline 562 & 1600 \\ & 1124 \\ \hline 5648 & 47600 \\ & \underline{45184} \end{array}$$

Find the roots of these equations to the nearest hundredth:

21. $x^2 - 4x - 7 = 0$

22. $5x^2 - 3x - 4 = 0$

23. $4 - x^2 = 7x$

24. $x^2 - .5x + .01 = 0$

25. $.08x^2 - 3x - .03 = 0$

26. $3x^2 - .42x = 1.46$

27. $x^2 - 3x + 1 = 0$

28. $x^2 = 1 - x$

29. $x^2 - 5x = 2$

30. $x + 5 = x^2$

31. $x = \frac{8}{x - 5}$

32. $x - 2 = \frac{2x}{x + 2}$

33. $\frac{5}{x - 2} + \frac{3}{x} = 4$

34. $\frac{1}{x - 2} + \frac{1}{x - 3} = 2$

35. $x = \frac{7}{x + 5}$

36. $\frac{x + 1}{x - 1} + \frac{x + 3}{x - 2} = 5$

How to solve quadratic equations graphically. In Chapter 9, we learned how to estimate the roots of first degree equations from their graphs. In much the same way, we can find the roots of quadratic equations or of higher equations from their graphs.

Illustration: Solve $x^2 - 4x - 5 = 0$ graphically.

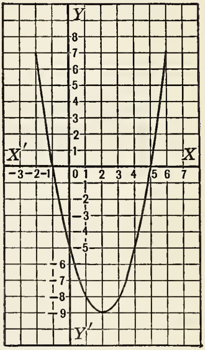
We shall first draw a graph of $y = x^2 - 4x - 5$ by substituting values of x and finding the corresponding values of y.

When $x = 1$, $y = 1^2 - 4 \cdot 1 - 5 = -8$
When $x = 2$, $y = 2^2 - 4 \cdot 2 - 5 = -9$

Making a table of these values, we have:

x	-2	-1	0	1	2	3	4	5	6
y	7	0	-5	-8	-9	-8	-5	0	7

Now we plot these points and draw a smooth curve through them, as shown in the figure.



This curve is called a **parabola.**

To solve the equation $x^2 - 4x - 5 = 0$ from the graph of $x^2 - 4x - 5 = y$, we notice first that y must equal 0. Now all points whose y coördinates are 0 lie on the x-axis, so evidently the roots will be the x values of the points in which our curve $x^2 - 4x - 5 = y$ cuts the x-axis. These values, -1 and $+5$, are then the roots of our quadratic equation. From this graph we can see more clearly why a quadratic equation has two roots.

Thought questions. As x increases from -2 to $+6$, how does y change? For what values of x is y negative? How are these values related to the roots of the equation? What value of x gives the lowest point on the curve? How is this value related to the roots? Has this curve an axis of symmetry? What is the relation of this axis of symmetry to the line segment joining the points for which $x = -1$ and $x = +5$?

Exercises

Solve graphically:

1. $x^2 - 4x + 3 = 0$
2. $x^2 - 6x + 8 = 0$
3. $x^2 - 6x + 5 = 0$
4. $x^2 - 2x - 3 = 0$
5. $x^2 - x - 6 = 0$
6. $x^2 + 6x + 8 = 0$
7. $x^2 + 5x + 4 = 0$
8. $x^2 - 3x = 0$
9. $x^2 - 2x = 8$
10. $x^2 + 2x = 3$
11. $2x^2 + x - 6 = 0$
12. $3x^2 - 2x - 5 = 0$
13. $2x^2 + 3x - 5 = 0$
14. $6x^2 - 5x - 6 = 0$
15. $5x^2 - x - 6 = 0$
16. $x^2 - 9 = 0$

The parabola in war. What will the bomb hit? If an aviator drops a bomb when he is directly over his target, will he hit the target, or should he have dropped it some time before arriving overhead?

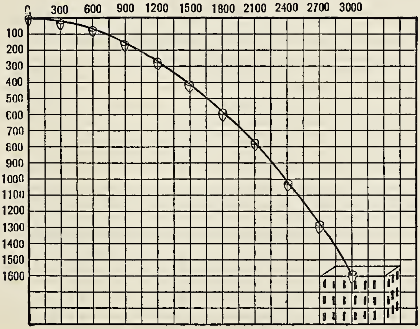
As the plane is moving forward, the bomb is moving horizontally with the speed of the plane, say 300 ft. a sec. But it is also pulled down by gravity a distance d, given by the formula

$$d = 16\,t^2$$

where d is the distance in feet and t the time in seconds.

Let us make a table of values for t and d.

t	1	2	3	4	5	6	7	8	9	10
Horizontally	300	600	900	1200	1500	1800	2100	2400	2700	3000
Vertically	16	64	144	256	400	576	784	1024	1296	1600



The Quadratic Equation in Geometry

1. Find the dimensions of a rectangle whose area is 104 if its base is 5 more than its altitude.

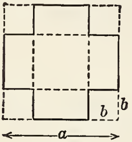
2. Find the side of a square whose area is doubled when the side is increased by 3.

3. A square rug is 2 ft. from each wall of a room and covers half the floor. Find the length of the rug to the nearest tenth of a foot.

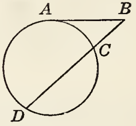
4. The base of a triangle exceeds the altitude by 4, and the area is 48. Find the altitude.

5. A square piece of tin is to have small squares cut from its corners and the sides bent up to make a pan 2 in. deep. What dimensions should the piece of tin have at first if the pan is to hold 98 cu. in.?

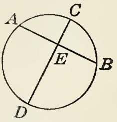
6. If AB is a tangent to the circle, it is found that the square of AB equals the product of BC and BD. If AB is 6, and CD exceeds BC by 1, find BC.



Ex. 5



Ex. 6



Ex. 7

7. When two chords, AB and CD intersect in a circle, the product of the segments, $AE \times EB$ and $CE \times ED$, are always equal. If AE and EB are each 4, and if the total length of CD is 10, find the segments of CD.

8. The area of a field is 96 sq. rds. What are its dimensions if it requires 40 rds. of fence to enclose it?

9. The sides of an equilateral triangle are increased respectively 1, 3, and 5. If it then becomes a right triangle, what was the side of the equilateral triangle? Remember that in a right triangle $a^2 + b^2 = c^2$, where a, b, and c are the sides.

10. The diagonal of a square is 1 in. longer than a side. Find the side of the square to the nearest tenth.

11. The length of a picture is to be 20 in. Artists believe that the picture would be the most pleasing if the width w were such that $\frac{20}{w} = \frac{w}{20 - w}$. To the nearest tenth of an inch, what should be the width of the picture?

12. Using the formula in exercise 11, find the width of a picture whose length is 4 in.

The Quadratic Equation in Science

1. If a ball is shot upward at a speed of 128 ft. a sec., the height it will reach in t sec. is given by the formula $128\,t - 16\,t^2$. After how many seconds will it have risen 256 ft.?

2. When an aviator is traveling over sea or level ground, the greatest distance on the earth that he can see is given by the formula $d^2 = 8000\,h + h^2$, where h is his height in miles. How high must he ascend to see 100 mi.?

3. The height of a circular arch is given by the formula $4\,h^2 - 8\,hr + s^2 = 0$, where h is the height, r the radius, and s the span. How high must an arch be built to have a span of 32 ft. and a radius of 20 ft.?

EQUATIONS LEADING TO QUADRATICS

Sets of two equations. You have found that it is often easier to solve a problem by using two unknowns than by using only one. Sometimes one of these equations is of the second degree, that is, contains a letter squared or the product of two letters, so that it is necessary to solve a quadratic equation.

How to solve sets of equations in which one is of the second degree.

Illustration: Solve:

$$\begin{aligned} x^2 + y^2 &= 10 \\ 2\,x + y &= 7 \end{aligned}$$

1. Solve the first-degree equation for one letter. It is better in this example to solve for y, since it has no coefficient and we can avoid fractions:

$$y = 7 - 2\,x$$

2. Substitute this value of y in the second-degree equation:

$$x^2 + (7 - 2x)^2 = 10$$

3. Now solve this equation:

$$\begin{aligned} x^2 + 49 - 28x + 4x^2 &= 10 \\ 5x^2 - 28x + 39 &= 0 \end{aligned}$$

$$\begin{array}{cc} 5x - 13 = 0 & x - 3 = 0 \\ x = \frac{13}{5} & x = 3 \end{array}$$

4. Substitute these values in the first-degree equation so as to find y, and group your answers:

$$\begin{array}{l|l} \text{When } x = \frac{13}{5} & \text{When } x = 3 \\ y = 7 - \frac{26}{5} & y = 7 - 6 \\ \quad = \frac{9}{5} & \quad = 1 \end{array}$$

Answers

x	3	$\frac{13}{5}$
y	1	$\frac{9}{5}$

5. Check in both equations:

$$\begin{array}{rcl|rcl} (\frac{13}{5})^2 + (\frac{9}{5})^2 & \vert & 10 & 3^2 + 1^2 & \vert & 10 \\ \frac{169}{25} + \frac{81}{25} & \vert & 10 & 9 + 1 & \vert & 10 \\ 10 & = & 10 & 10 & = & 10 \\ \frac{26}{5} + \frac{9}{5} & \vert & 7 & 6 + 1 & \vert & 7 \\ 7 & = & 7 & 7 & = & 7 \end{array}$$

Exercises

Solve and check:

1. $x + y = 5$
$x^2 + y^2 = 13$

2. $xy = 15$
$x + y = 8$

3. $x^2 - y^2 = 24$
$2x + y = 11$

4. $xy + y^2 = 8$
$x - 2y = 10$

5. $a^2 + 2b^2 = 11$
$2a + b = 5$

6. $c - d = 1$
$5c^2 - 3d = 17$

7. $M^2 + MN = -10$
$3M + N = 8$

8. $w_1^2 + w_2^2 = 13$
$w_1 - w_2 = 1$

9. $x^2 + xy + y^2 = 4$
$x - y = 2$

10. $d = 16t^2$
$d + t = 66$

Equations solved by factoring.

Illustration: Solve: $2x^5 - 20x^3 + 18x = 0$

Factoring:

$$\begin{aligned} 2x(x^4 - 10x^2 + 9) &= 0 \\ 2x(x^2 - 9)(x^2 - 1) &= 0 \\ 2x(x+3)(x-3)(x+1)(x-1) &= 0 \end{aligned}$$

Setting each factor equal to 0:

$$\begin{aligned} 2x &= 0 \qquad & x &= 0 \\ x + 3 &= 0 & x &= -3 \\ x - 3 &= 0 & x &= +3 \\ x + 1 &= 0 & x &= -1 \\ x - 1 &= 0 & x &= +1 \end{aligned}$$

Exercises

Solve by factoring:

1. $x^2 - 3x = 0$
2. $x^2 = 5x$
3. $x^3 - 4x = 0$
4. $x^3 = 9x$
5. $7x^2 - 28 = 0$
6. $3x^2 = 48$
7. $x^4 - x^2 = 0$
8. $x^3 + 2x^2 = 35x$
9. $3x^3 + 12x = 15x^2$
10. $4x^5 = x^3$
11. $x^4 - 10x^2 + 9 = 0$
12. $4x^4 - 37x^2 = -9$
13. $3x^4 - 15x^2 + 12 = 0$
14. $x^4 - 2x^2 + 1 = 0$
15. $17x^2 = x^4 + 16$
16. $4x^4 + 1 = 5x^2$
17. $5x^5 - 25x^3 + 20x = 0$
18. $3x^5 + 3x = 6x^3$
19. $x^{\frac{2}{3}} - 5x^{\frac{1}{3}} + 6 = 0$
20. $x^{\frac{1}{2}} - 3x^{\frac{1}{4}} + 2 = 0$

RADICALS

One day Dorothy did these square-root exercises for home work:

$$\begin{aligned} \sqrt{2} &= 1.4142 \qquad & \sqrt{3} &= 1.732 \\ \sqrt{8} &= 2.8284 & \sqrt{12} &= 3.464 \\ \sqrt{18} &= 4.2426 & \sqrt{27} &= 5.196 \end{aligned}$$

Katherine pointed out that the square root of 8 was exactly twice as large as the square root of 2, and that the square root of 18 was just 3 times as large. Dorothy then noticed that the same relation held for the $\sqrt{3}$, $\sqrt{12}$, and

$\sqrt{27}$. She remarked, "What a lot of work I could have saved myself if I had only known this in advance."

Now there are many square roots that can be obtained from others by a simple multiplication. Can you discover the secret for yourself so you can avoid all the labor of extracting these square roots?

Let us examine these radicals. Since $8 = 4 \times 2$ and 4 is a perfect square, $\sqrt{8} = \sqrt{4 \times 2} = 2\sqrt{2}$, for we can take the 4 out of the radical. Similarly $\sqrt{18} = \sqrt{9 \times 2} = 3\sqrt{2}$; so you see why $\sqrt{8}$ and $\sqrt{18}$ were 2 and 3 times the $\sqrt{2}$.

This process of taking a factor that is a square out of the radical is called ***simplifying radicals.*** Of course if we had a cube root instead of a square root, we should have to take out a factor that was a cube.

Illustration. Simplify $\sqrt{75}$.

The **largest** factor of 75 that is a square is 25. Then $\sqrt{25} = 5$

$$\sqrt{75} = \sqrt{25 \times 3}$$
$$= 5\sqrt{3}$$

Class Exercises

Simplify:

1. $\sqrt{12}$
2. $\sqrt{20}$
3. $\sqrt{32}$
4. $\sqrt{28}$
5. $\sqrt{40}$
6. $\sqrt{44}$
7. $\sqrt{45}$
8. $\sqrt{98}$
9. $\sqrt{24}$
10. $\sqrt{48}$
11. $\sqrt{96}$
12. $\sqrt{54}$
13. $\sqrt[3]{16}$
14. $\sqrt[3]{24}$
15. $\sqrt[3]{40}$
16. $\sqrt[3]{54}$
17. $\sqrt[3]{32}$
18. $\sqrt[4]{80}$
19. $2\sqrt{12}$
20. $5\sqrt{27}$

Optional Exercises

21. $\sqrt[3]{-32}$
22. $\sqrt[3]{135}$
23. $\sqrt{.18}$
24. $\sqrt{1.8}$
25. $\sqrt{2.5}$
26. $\sqrt[3]{.08}$
27. $\sqrt{10.8}$
28. $\sqrt{.24}$

If $\sqrt{2} = 1.4142$, $\sqrt{3} = 1.732$, *and* $\sqrt{5} = 2.236$, *find correct to three significant figures:*

29. $\sqrt{50}$
30. $\sqrt{20}$
31. $\sqrt{27}$
32. $\sqrt{75}$
33. $\sqrt{128}$
34. $\sqrt{98}$
35. $\sqrt{80}$
36. $\sqrt{48}$

Honor Work

Simplify:

37. $\sqrt{5a^2}$

38. $\sqrt{18a^3}$

39. $\sqrt{ab^3c^2}$

40. $\sqrt{x^{17}}$

41. $5a\sqrt{9a^3}$

42. $\sqrt[3]{24x^3y^5}$

43. $2m\sqrt{20m^7n^4}$

44. $\sqrt[4]{32x^5y^6z^2}$

How to get a radical out of a denominator or a denominator out of a radical. They are really the same thing under two names, for $\frac{\sqrt{2}}{\sqrt{3}}$ is equal to $\sqrt{\frac{2}{3}}$.

Illustration 1.

Find to four decimal places the value of $\frac{4}{\sqrt{2}}$.

Since $\sqrt{2} = 1.41421$, we obtain $4 \div 1.41421$. By long division this gives 2.8284. This division is hard work. Can we find an easier way? If we could first get the radical out of the denominator, we would not have to divide by this long decimal. And here is the way to do it.

Multiply both numerator and denominator by $\sqrt{2}$:

$$\frac{4}{\sqrt{2}} = \frac{4\sqrt{2}}{\sqrt{2}\sqrt{2}}$$
$$= 2\sqrt{2}$$

Illustration 2. Simplify: $\sqrt{\frac{5}{3}}$

Multiply both numerator and denominator by 3:

$$\sqrt{\frac{5}{3}} = \sqrt{\frac{15}{9}}$$
$$= \tfrac{1}{3}\sqrt{15}$$

Exercises

Simplify:

1. $\sqrt{\frac{2}{3}}$

2. $\sqrt{\frac{3}{7}}$

3. $\sqrt{\frac{1}{3}}$

4. $\sqrt{\frac{3}{8}}$

5. $\sqrt{\frac{4}{5}}$

6. $\sqrt{\frac{3}{4}}$

7. $\sqrt{\frac{9}{2}}$

8. $\sqrt{\frac{18}{5}}$

9. $\frac{\sqrt{8}}{\sqrt{3}}$

10. $\frac{\sqrt{5}}{\sqrt{2}}$

11. $\frac{\sqrt{3}}{\sqrt{18}}$

12. $\frac{1}{\sqrt{20}}$

13. $\frac{5\sqrt{7}}{\sqrt{5}}$

14. $\frac{3\sqrt{2}}{\sqrt{8}}$

15. $\frac{\sqrt{27}}{\sqrt{32}}$

16. $\frac{\sqrt{9}}{\sqrt{50}}$

PUZZLE PROBLEMS

Coin Puzzles

I have 16 coins, nickels and dimes, and they are worth \$1. How many nickels have I?

Kind of Coin	*Number*	*Value in Cents*
5¢	n	$5n$
10¢	d	$10d$

$$\begin{aligned} n + d &= 16 \\ 5n + 10d &= 100 \\ n &= 12 \\ d &= 4 \end{aligned}$$

Check:
12 nickels = 60¢
4 dimes = 40¢
16 coins = 100¢

Solve and check:

1. Paul has 18 coins, dimes and quarters, worth \$2.55. How many of each has he?

2. Thirty coins, nickels and pennies, are worth 50¢. How many of each kind of coin are there?

3. John has \$3.11 in pennies, nickels, and quarters. He has 4 more nickels than pennies, and 2 less quarters than pennies. How many pennies has he?

4. Twenty-eight coins are worth \$3.15. The number of dimes equals the number of quarters, and the number of nickels equals their sum. How many dimes are there?

5. Twenty-four coins are worth \$2.60. There are 3 more dimes than quarters and 3 more nickels than dimes. How many quarters are there?

6. Robert had \$2.30 in dimes and quarters. He paid a debt of 5¢ with a quarter and received dimes in change. Now he has twice as many dimes as quarters. How many of each had he at first?

Motion Puzzles

1. Henry can row 3 mi. an hr. in still water. If he rows on a river that flows 1 mi. an hr., what is his rate up-stream? His rate down-stream? What did you do to the 3 and 1 to get the rate up-stream? To get the rate down-stream?

2. An aviator can fly s mi. an hr. when there is no wind. The wind is now blowing w mi. an hr. (*a*) What is his rate with the wind? Against the wind? (*b*) How long will it take him to go 100 mi. with the wind? Against the wind? (*c*) How far can he fly in 5 hrs. with the wind? Against the wind?

3. A motor boat goes 45 mi. up-stream in 5 hrs. and returns down-stream in 3 hrs. What is its speed in still water?

Illustration: Let b = the no. of mi. an hr. the boat can go in still water

c = the no. of mi. an hr. the current moves

	Rate	*Time*	*Distance*
Up-stream........	$b - c$	5	45
Down-stream.....	$b + c$	3	45

$$5(b - c) = 45$$
$$3(b + c) = 45$$
$$b = 12 \text{ mi. an hr.}$$
$$c = 3 \text{ mi. an hr.}$$

Check: Rate up $= 12 - 3 = 9$

Rate down $= 12 + 3 = 15$

$$R \times T = D$$
$$9 \times 5 = 45$$
$$15 \times 3 = 45$$

4. Dorothy rowed 3 mi. down-stream in 1 hr., but it took her 3 hrs. to return. What is her rate is still water, and what is the rate of the stream?

5. Mr. Byrd flew 600 mi. against the wind in 4 hrs. and then returned in 3 hrs. What speed could he make in still air?

6. A steamer can travel 12 mi. an hr. in still water. It goes 2 hrs. down-stream and finds that it requires 4 hrs. to return. What is the rate of the current?

7. A plane travels 200 mi. an hr. In 5 hrs. how far may it go against a 40 mi. an hr. wind and return with the wind?

Number Puzzles

1. Sarah said, "Think of a number. Add 10. Multiply by the original number. Now add 25. What is your answer?"

(*a*) Emily answered, "49." What number did she start with?

(*b*) Tom replied, "81." What was his number?

2. Marietta asked, "What number added to its reciprocal gives 2?" "What's a reciprocal?" asked Florence. "The reciprocal of a number is 1 divided by the number. For example, the reciprocal of 3 is $1 \div 3$, or $\frac{1}{3}$, and the reciprocal of $\frac{2}{5}$ is $1 \div \frac{2}{5}$ or $\frac{5}{2}$." Solve Marietta's puzzle.

3. What number leaves a remainder of $\frac{7}{12}$ when subtracted from its reciprocal?

4. Find two consecutive numbers whose product is 72.

5. The sum of the reciprocals of two consecutive odd numbers is $\frac{12}{35}$. What are the numbers?

6. In our classroom, the number of seats in a row is 2 more than the number of rows, and the room seats 35 pupils. How many rows are there?

Geometry Puzzles

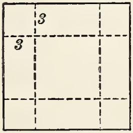
1. The length of a rectangle is 3 more than its width, and its area is 40. Find its dimensions.

2. The area of a rectangle is 72, and its perimeter is 34. Find its dimensions.

3. The base of a triangle is 5 more than its altitude, and its area is 18. Find its altitude.

4. The perimeter of a right triangle is 30, and its hypotenuse is 13. Find its other two sides.

5. I want to make an open box 3 in. deep by cutting squares from the corners of a square piece of tin and folding it along the dotted lines. If the box must contain 75 cu. in., what dimensions must the original square of tin have?



6. A rectangular piece of tin is twice as long as it is wide. Charles cut a 2 in. square from each corner and folded up the sides to make a box that would hold 60 cu. in. What were the dimensions of the piece of tin?

7. The dimensions of a picture are in the ratio 5:3. Its frame is 3 in. wide, and the area of the framed picture is 315 sq. in. What are the dimensions of the picture?

8. Two automobilists leave a town at the same time on roads that make a right angle with each other. Their rates are as 3:4, and in 2 hrs. they are 100 mi. apart. What are their rates?

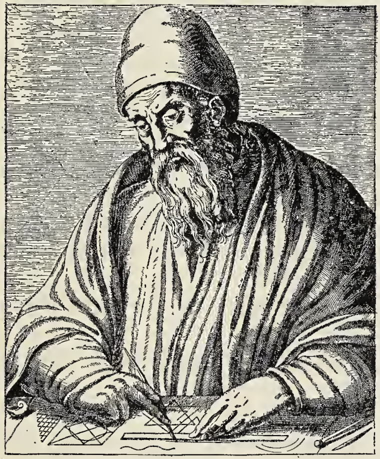
Chapter 17

DEMONSTRATIVE GEOMETRY

Why we study demonstrative geometry. Reasoning. The power to reason is man's most valuable possession, and the one which enables him to surpass the horse or dog or any other animal. The ability to weigh facts and to draw the correct conclusions is one well worth developing, for in any form of activity, situations are continually arising on which judgments must be formed. For example, the business man collects all the data he can get about the enterprise he intends to undertake. These he studies and from them draws certain conclusions. His success or failure depends largely on his ability to draw correct conclusions. Again, the physician notes the symptoms of a patient and from them diagnoses the case. On the accuracy of his decision life may depend. Even in the small things of life—whether at work or at play—decisions must be made, and judgments formed.

Geometry is a study in reasoning. Certain facts are given us. From these we draw conclusions. We prove that a statement is true if the data given are true. To make sure that we shall not slip in our reasoning, we follow a definite form, giving a reason for every step in the process.

Geometry deals with points, lines, angles, circles, and figures made from them, such as triangles. In the earlier chapters of this book, we have used many geometric figures, such as vertical angles, parallel lines, triangles, similar figures, etc. Before beginning to prove statements, then, let us first review what we have learned already.



EUCLID

Euclid was a famous Greek mathematician who lived in Alexandria, Egypt, about 300 B.C. Little is known about his life, his reputation depending entirely on his mathematical works. He wrote the world's first great textbook. Before his time many discoveries had been made in geometry, particularly by the Pythagoreans, but this knowledge was not organized or systematized until Euclid undertook the task. He did his work so well that his book was used as a text until recent times and more editions of it were issued than of any other textbook written.

Facts you already know. 1. A ***point*** is represented by a dot. The smaller the dot is, the nearer it is to a point, for a geometric point has no size. We read a point by a capital letter placed near it, as point A.

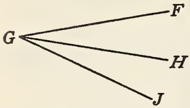
2. A ***straight line*** resembles a stretched thread. The thinner the thread is, the nearer it is to a line, for a geometric line has no width or thickness. It has only length.

A geometric line does not have ends, but goes on and on as far as you can think. A ***line segment*** is a part of a line and has ends.

3. A ***circle*** is a closed plane curve, all points of which are the same distance from a fixed point in the plane called the ***center***. Any part of the circle is an ***arc***. The ***radius*** is a line from the center to any point on the curve. Therefore, all radii of a circle are equal. A ***chord*** is a line joining two points of the circle. A chord passing through the center is a ***diameter***.

4. An ***angle*** ($\angle$) is formed by two straight lines that meet at a point. The lines are the ***sides*** of the angle, and the point is its ***vertex***.

5. ***Adjacent angles*** are two angles that have the same vertex and a common side between them, as $\angle FGH$ and $\angle HGJ$.



6. A ***straight angle*** is an angle whose sides lie in a straight line in opposite directions from the vertex.

7. If one straight line meets another straight line so as to make two adjacent angles equal, each of these angles is a ***right angle.***

8. Two lines meeting at right angles are ***perpendicular*** to each other.

9. An ***acute angle*** is an angle smaller than a right angle, and an ***obtuse angle*** is an angle larger than a right angle, but smaller than a straight angle.

10. When two straight lines cross, the opposite angles are ***vertical angles.***

11. ***Supplementary angles*** are two angles whose sum is a straight angle.

12. All right angles are equal.

13. If two adjacent angles have their exterior sides in a straight line, they are supplementary.

14. Supplements of an angle are equal.

15. Vertical angles are equal.

Review Exercises

1. Draw two unequal line segments. Call the longer a and the shorter b. Construct a segment equal to:

(*a*) $3a$
(*b*) $2b$
(*c*) $a + b$
(*d*) $a - b$
(*e*) $\frac{1}{2}a$
(*f*) $\frac{3}{2}b$
(*g*) $3a + b$
(*h*) $3a - 2b$

2. Draw two unequal angles—$\angle A$ about 40° and $\angle B$ about 25°. Construct an angle equal to:

(*a*) $2A$
(*b*) $3B$
(*c*) $A + B$
(*d*) $A - B$
(*e*) $\frac{1}{2}A$
(*f*) $\frac{1}{4}B$
(*g*) $2\frac{1}{2}A$
(*h*) $3A - B$

3. Find the supplement of each of these angles:

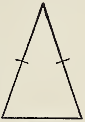
(*a*) 15°
(*b*) 128°
(*c*) 42°
(*d*) 90°
(*e*) 156°
(*f*) 34°
(*g*) 62°40′
(*h*) 146°23′

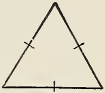
4. Construct an angle of:
(*a*) 90° (*b*) 45° (*c*) 135° (*d*) 22°30′

5. Through how many degrees does the earth turn in:
(*a*) 24 hrs.? (*b*) 5 hrs.? (*c*) 40 min.?

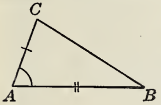
TRIANGLES

Facts you already know. 1. A *triangle* (△) is a figure formed by three straight lines that enclose a part of the plane.

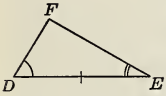

Isosceles Triangle


Equilateral Triangle

2. An *isosceles triangle* is a triangle having at least two of its sides equal. The two equal sides are the ***legs,*** and the angle formed by them is the ***vertex angle.*** The third side is the ***base,*** and the angles adjoining it are the ***base angles.***



3. An *equilateral triangle* is a triangle having all three sides equal.



When two sides of a triangle are mentioned, the angle formed by these two sides is called the ***included angle.*** In the figure, ∠*A* is included by *AB* and *AC*.

When two angles of a triangle are mentioned, the side joining their vertices is called the ***included side.*** *DE* is included by ∠*D* and ∠*E*.

CONGRUENT TRIANGLES

Two triangles are congruent (≅) if one of them can be placed so that every part of it will fit exactly on the corresponding part of the other.

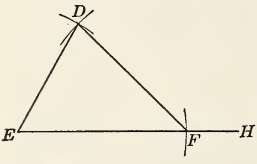
Corresponding sides and corresponding angles of congruent triangles are equal. This is one of the most important ways of proving that a line equals another line or that an angle equals another angle.

How to copy a triangle. Here are three methods of constructing a triangle congruent to a given triangle, ABC.

First Method. 1. On the line EH, measure a length EF equal to BC.

2. With E as center and BA as radius, draw an arc.

3. With F as center and CA as radius, draw another arc cutting the last one at D.



4. Draw DE and DF.

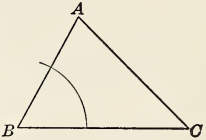
Do you think that $\triangle DEF$ is congruent to $\triangle ABC$? How many sides of $\triangle ABC$ did we measure? Do you think we would always get a triangle congruent to $\triangle ABC$ if we make its three sides equal to those of $\triangle ABC$?

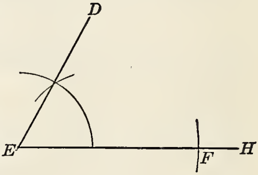
Second Method. 1. On EH measure a length EF equal to BC.

2. At E make an angle equal to $\angle B$.

Now how can we finish the triangle? If we measure a length from E on ED equal to BA and complete the triangle, do you think it would be congruent to $\triangle ABC$?

How many sides did you measure? How many angles? Is the angle that you measured included by the sides that you measured?

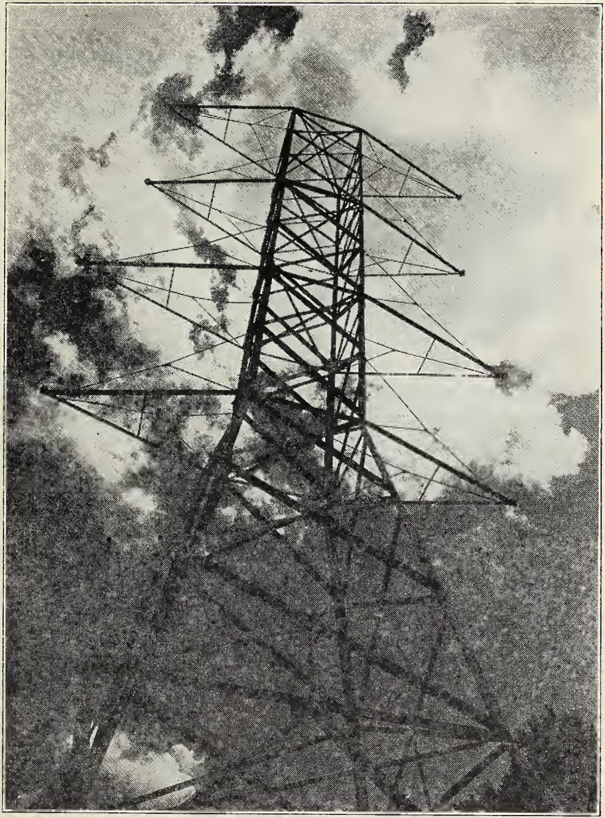




Do you think that a triangle would always be congruent to $\triangle ABC$ if you made two of its sides and their included angle equal to those of $\triangle ABC$?

Third Method. After you had made EF equal to BC and $\angle E$ equal to $\angle B$ in Method 2, could you have completed the work differently? If you made an angle at F equal to $\angle C$ and extended the line until it crossed ED, do you think your triangle would be congruent to $\triangle ABC$?

How many sides did you measure? How many angles? Is the side included by the angles? Do you think that a triangle



Courtesy of Tennessee Valley Authority.

TRANSMISSION TOWER

Notice the similar and congruent triangles which have been utilized by engineers in the construction of this tower.

would always be congruent to $\triangle ABC$ if you made two angles and their included side equal to those of $\triangle ABC$?

If you have constructed these triangles correctly, you have discovered that:

1. Two triangles are congruent, if two sides and the included angle of one equal respectively two sides and the included angle of the other. (*s.a.s.*)

2. Two triangles are congruent, if two angles and the included side of one equal respectively two angles and the included side of the other. (*a.s.a.*)

3. Two triangles are congruent if the three sides of one equal the three sides of the other. (*s.s.s.*)

Exercises

1. Draw an angle of about 70°. Now construct a triangle having two sides respectively 2 in. and 3 in. and this angle as their included angle.

2. Draw an angle of about 100°. Repeat the above construction using this angle.

3. Construct a right angle. Then construct an isosceles triangle having this angle as vertex angle and a leg equal to 3 in.

4. Can you construct a right triangle (*a*) Which has all sides unequal? (*b*) Which is isosceles? (*c*) Which is equilateral?

5. Construct angles of 90° and 45°. Then construct a triangle having a base 3 in. long included by these angles.

6. Construct a right triangle whose legs are $1\frac{1}{2}$ in. and 2 in.

7. Construct an isosceles triangle and bisect its vertex angle. Is this line perpendicular to the base? Does it bisect the base?

8. Construct a triangle that is not isosceles and bisect its vertex angle. Is this line perpendicular to the base? Does it bisect the base?

9. Can you construct a triangle whose sides are:

(*a*) 2 in., 3 in., 4 in.? (*c*) 2 in., 3 in., 6 in.?
(*b*) 2 in., 3 in., 5 in.? (*d*) 2 in., 3 in., 2 in.?

THE THEOREM

In geometry, we prove that a statement is true *if* the conditions given are true. This statement is called a theorem. It consists of two parts—(1) a part that you are to accept as a starting point called the "given," and (2) the part you are to prove. The given part is usually either a clause beginning with "if," or it is the subject of the sentence. In the latter case, the predicate is the part to be proved. When there is an "if" clause, the independent clause, that not containing the "if," is the part to be proved.

Of course, if the part given is not true, the conclusion "proved" may not be true. This is the way many advertisers fool people. They begin with a false assumption and then make statements that *would* be true *if* the assumption had been true. You must therefore examine carefully the statements assumed as facts before you believe the conclusions drawn from them.

Read some advertisements, and think over whether their "facts" are necessarily true. Then see if the conclusion drawn would follow from these "facts."

Most arguments result from a disagreement as to the facts to begin with. One says that a thing is true; the other denies it; and they get nowhere. If you wish to convince a person, be sure to start with something that he will admit is true. Then be sure that you have correct reasons for your conclusions.

AXIOMS

If we are to give a reason for every step in a proof, we must agree on some facts that we can give as reasons. Let us list a few such statements, most of which you have already used. They are called ***axioms.***

The axioms.

1. In any process a quantity may be substituted for an equal one (called "substitution").

2. If equals are added to equals, the results are equal.

3. If equals are subtracted from equals, the results are equal.

4. If equals are multiplied by equals, the results are equal. (Special case: Doubles of equals are equal.)

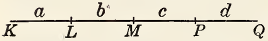
5. If equals are divided by equals, the results are equal. (Special case: Halves of equals are equal.)

6. The whole equals the sum of all its parts.

Exercises

1. If $a = b$ and $b = c$, why does $a = c$?

2. If $a = b$, $c = d$, and $b = d$, why does $a = c$?



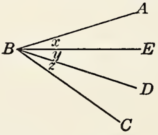
3. If $a = c$ and $b = d$, why does $a + b = c + d$?

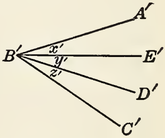
4. If $a = c$ and $b = d$, why does $a - b = c - d$? Why does $LP = MQ$?

5. If $a = c$, why does $a + b = b + c$?

6. If $KM = LP$, why does $a = c$?

7. If $KM = MQ$ and $a = d$, why does $b = c$?





8. If $\angle x = \angle z$, why does $\angle ABD = \angle EBC$?

9. If $\angle ABD = \angle EBC$, why does $\angle x = \angle z$?

10. If $\angle B$ and $\angle B'$ are each cut into three equal angles, x, y, z and x', y', z' respectively, and $\angle x = \angle x'$, why does $\angle B$ equal $\angle B'$?

Give an axiom for each of these exercises:

11. If $x + 4 = 9$, then $x = 5$.

12. If $x - 7 = 5$, then $x = 12$.

13. If $3x = 6$, then $x = 2$.

14. If $\frac{1}{5}x = 3$, then $x = 15$.

15. If $5x - 4 = 2x + 5$ and $x = 3$, then $15 - 4 = 6 + 5$.

16. Solve for x, giving an axiom for each step in the solution:

(*a*) $5x + 13 = 34 - 2x$

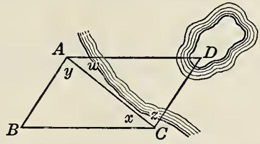
(*b*) $\frac{x}{2} - \frac{x}{5} = 12$

(*c*) $3x - y = 3$
$y = 2x$

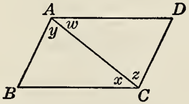
(*d*) $x - 4.2 = .4x$

THE DIRECT PROOF

To find the distance from A on the shore to an island D, Robert used a transit and made $\angle w$ equal to $\angle x$, and $\angle y$ equal to $\angle z$. Then he measured the distance from B to C. John would not believe that this was necessarily the correct distance from A to D, so Robert proved it as follows.



Notice that the "given," that Robert started with, included only those facts that John knew were true. Also notice that he gives a good reason for every statement he makes in his proof.



Robert's Proof

Given: $\angle x = \angle w$ and $\angle y = \angle z$.

To prove: $BC = AD$.

Proof:

STATEMENTS	REASONS
1. In $\triangle s$ ABC and ADC, $AC = AC$.	1. A line equals itself.
2. $\angle x = \angle w$.	2. Given.
3. $\angle y = \angle z$.	3. Given.
4. $\triangle ABC \cong \triangle ADC$.	4. Two triangles are congruent if two $\angle s$ and the included side of one equal two $\angle s$ and the included side of the other.
5. $BC = AD$.	5. Corresponding sides of congruent triangles are equal.

"But," asked John, "how do you know they are corresponding sides?" "Well," answered Robert, "AD is the side opposite $\angle z$, and BC is opposite its equal, $\angle y$. And the sides opposite equal angles are the corresponding sides."

Arrangement of proof. 1. At the top of your paper, you should write out in words the theorem to be proved. Robert omitted this.

2. Next draw a neat figure with ruler and pencil.

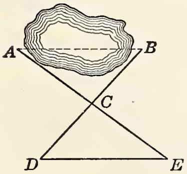
3. State what is given in terms of the figure, that is, write "$\angle x = \angle y$" and not simply "two angles are equal."

4. State what you are to prove.

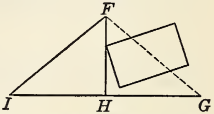
5. Draw a straight line down through the middle of the page. On the left of this line, write and number the statements. On the right of the line, opposite each statement, write its reason, numbered to correspond with the statement.

Exercises

1. To find the distance across a pond from A to B, Robert Young set up a pole at C and measured a distance CD equal to BC, and a distance CE equal to AC. Show that he can find the required distance by measuring DE.

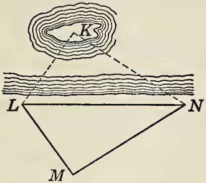


2. Engineers for the Lively Traffic R. R. Co. are surveying a right of way through John Greer's farm. At one place their work is obstructed by a barn. They want to know the distance from F to G. They lay off any line HG and make FH perpendicular to it. HI is then taken equal to HG. Show that FI is the same length as the distance from G to F.

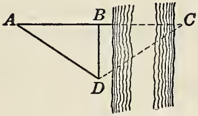


3. To measure the distance from a point L on the shore to an island K, a surveyor found that $\angle KLN$ equaled 40° and $\angle KNL$ equaled 35°. He then laid off a triangle LMN with $\angle NLM = 40°$

and ∠*LNM* = 35°. Show what line he had to measure to find the distance from *L* to *K*.



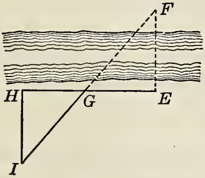
Ex. 3



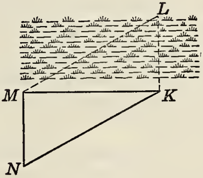
Ex. 4

4. To find the distance from *B* across a river to a point *C*, *BD* is drawn perpendicular to *BC*. At the point *D*, ∠*BDA* is taken equal to ∠*BDC*. What line is equal to *BC*? Prove your answer.

5. A man can find the distance *EF* across a river as follows. He measures *EH* at right angles with *EF*, and sets a stake at the middle point *G*. Then he measures the distance at right angles from *H* until he arrives at a point *I* at which he is in line with *G* and *F*. Show that *EF* equals the last distance he has measured.



Ex. 5



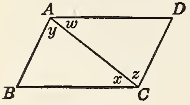
Ex. 6

6. Two boys, Henry and James, find a distance *KL* as follows. Henry walks at right angles to *KL* any distance *KM* and measures the ∠*KML* with his protractor. He calls back to James, who is at *K*, that the angle is 25°. The latter then measures an angle of 25°, ∠*MKN*, and travels along *KN* until Henry, who is at *M*, observes that he is at right angles with *MK*. They decide that *MN* equals *KL*. Prove that they are right.

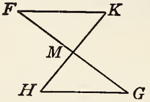
7. If $AD = BC$ and $\angle w = \angle x$, prove that $\triangle ABC \cong \triangle ACD$.

8. If $AD = BC$ and $DC = AB$, prove that $\triangle ABC \cong \triangle ACD$.

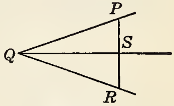
9. If the lines FG and KH bisect each other at M, prove $\triangle FKM \cong \triangle GHM$.



Exs. 7, 8



Ex. 9

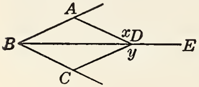


Ex. 10

10. If QS bisects $\angle PQR$, and $PR \perp QS$ prove $\triangle PQS \cong \triangle RQS$.

11. If $\angle ABC$ is bisected by BE, and $AB = BC$, prove $\triangle ABD \cong \triangle CBD$.

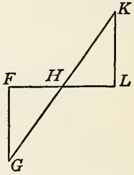
12. If $\angle ABC$ is bisected by BE, and $\angle x = \angle y$, prove $\triangle ABD \cong \triangle CBD$.



13. If $FG \perp FL$ and $KL \perp FL$, and KG passes through the middle point of FL, prove that $\triangle FHG \cong \triangle LHK$.

14. The bisector of the vertex angle of an isosceles triangle divides the figure into two congruent triangles.

15. Two right triangles are congruent, if a leg and the adjoining acute angle of one equal respectively a leg and the adjoining acute angle of the other.



16. Two isosceles triangles are congruent, if a leg and the vertex angle of one triangle equal a leg and the vertex angle of the other.

17. Two right triangles are congruent, if the legs of one equal respectively the legs of the other.

18. If the bisector of an angle of a triangle is perpendicular to the opposite side, the triangle is isosceles.

19. If two opposite angles of a four-sided figure (quadrilateral) are bisected by the line joining their vertices, the quadrilateral has two pairs of equal sides.

20. Prove the construction for bisecting an angle.

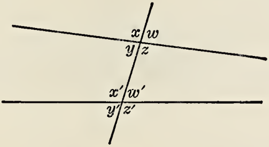
PARALLEL LINES

Facts you already know. 1. *Parallel* (||) *lines* are straight lines in the same plane that cannot meet however far extended in either direction.

2. A *transversal* is a line that cuts two or more other lines.

3. If two straight lines are cut by a transversal, the angles are named as follows:

∡ w', x', y, z, are ***interior*** ∡.
∡ w, x, y', z', are ***exterior*** ∡.



A pair of angles are ***alternate,*** when they are on opposite sides of the transversal, and one at each vertex, as ∡ y and w', z and x', w and y', and x and z'.

The pairs y and w', and z and x' are ***alternate interior angles.***

The pairs w and y', and x and z' are ***alternate exterior angles.***

The pairs w and w', x and x', y and y', and z and z', are ***corresponding angles.***

When alternate interior angles, or any of the above pairs of angles are mentioned, it is understood that there are two straight lines cut by a transversal.

4. Two lines are parallel if their alternate interior angles are equal.

5. Two lines are parallel if their corresponding angles are equal.

6. Alternate interior angles of parallel lines are equal.

7. Corresponding angles of parallel lines are equal.

Parallel line assumption. Through a point not more than one line can be drawn parallel to a given line.

A *quadrilateral* is a figure formed by four straight lines that enclose a part of the plane.

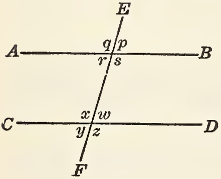
Exercises

Prove that:

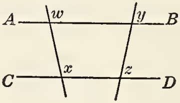
1. Lines perpendicular to the same line are parallel.

2. $AB \parallel CD$ if:
- (*a*) $\angle p = 65°$ and $\angle w = 65°$
- (*b*) $\angle p = 65°$ and $\angle x = 115°$
- (*c*) $\angle p = 70°$ and $\angle y = 70°$
- (*d*) $\angle p = 75°$ and $\angle z = 105°$
- (*e*) $\angle r = m°$ and $\angle x = 180° - m°$

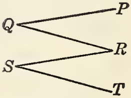
3. Two lines are parallel if their alternate exterior angles are equal.



Ex. 2



Ex. 5



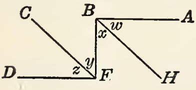
Ex. 6

4. Alternate exterior angles of parallel lines are equal.

5. If $\angle w = \angle x$, then $\angle y = \angle z$.

6. If $PQ \parallel RS$ and $QR \parallel ST$, then $\angle Q = \angle S$.

7. If $\angle w = \angle z$ and $\angle x = \angle y$, then $AB \parallel FD$.



8. If $AB \perp BF$ and $DF \perp BF$, BH bisects $\angle FBA$, and FC bisects $\angle BFD$, then BH and CF are parallel.

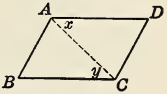
9. If GH and KL bisect each other, prove that $GK \parallel LH$.

10. A line perpendicular to one of two parallel lines is perpendicular to the other.

11. Lines parallel to the same line are parallel. (Draw a transversal and use corresponding angles.)

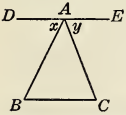
12. If the opposite sides of quadrilateral $ABCD$ are equal, they are parallel. Draw a line joining the opposite vertices of the quadrilateral.

13. If AD and BC are equal and $\angle x = \angle y$, then AB is parallel to DC.

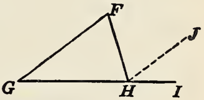


14. If the opposite sides of a quadrilateral are parallel, they are equal.

15. If two sides of a quadrilateral are both equal and parallel, the other two sides are equal and parallel.

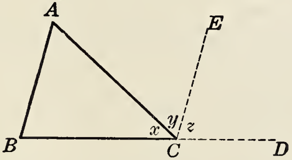


16. Prove that the sum of the angles of triangle ABC equals a straight angle.



17. Prove that the exterior angle FHI equals the sum of angles F and G.

The sum of the angles of a triangle is a straight angle.



Given: $\triangle ABC$.

To prove: $\angle A + \angle B + \angle x = 1$ straight $\angle$.

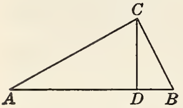
Proof:

Statements	Reasons
Extend BC to D and draw $CE \parallel BA$.	
1. $\angle x + \angle y + \angle z = 1$ st. $\angle$.	1. A st. $\angle$ is an $\angle$ whose sides lie in a st. line, etc.
2. $\angle y = \angle A$.	2. Alternate interior ∠s of parallel lines are equal.
3. $\angle z = \angle B$.	3. Corr. ∠s of parallel lines are equal.
4. $\angle x + \angle A + \angle B = 1$ st. $\angle$.	4. Substitution.

Exercises

1. If two triangles have two angles of one equal respectively to two angles of the other, their third angles are equal.

2. In $\triangle ABC$, find the number of degrees in $\angle C$ if:



(*a*) $A = 38°$ and $B = 74°$
(*b*) $A = B = C$
(*c*) $B = 48°$ and $A = C$
(*d*) $C = A + B$
(*e*) $A = m°$ and $B = n°$
(*f*) $C + A = 125°$ and $C + B = 130°$

3. If CD is perpendicular to the hypotenuse of right $\triangle ABC$, then $\angle ACD = \angle B$.

4. If the sum of two angles of a triangle equals the third angle, the triangle is a right triangle.

5. If two angles of a triangle are 30° and 60°, what angle is formed by their bisectors?

6. A triangle cannot have more than one right angle or more than one obtuse angle.

7. In $\triangle ABC$ find the number of degrees in each angle, if:

(*a*) $\angle B$ is twice $\angle A$, and $\angle C$ is 20° more than $\angle A$
(*b*) $\angle B$ is 3 times $\angle A$, and $\angle C$ is twice $\angle B$
(*c*) $\angle B$ exceeds $\angle A$ by 5°, and $\angle C$ exceeds $\angle B$ by 5°
(*d*) The sum of $\angle A$ and $\angle B$ is 130°, and 4 times $\angle A$ exceeds 5 times $\angle B$ by 16°
(*e*) $\angle B$ is twice $\angle A$, and $\angle C$ is $\frac{1}{2}$ the sum of $\angle A$ and $\angle B$.

8. Two triangles are congruent, if two angles and a side opposite one of them of one triangle equal two angles and the corresponding side of the other. Are the third angles equal? Why?

9. Two right triangles are congruent, if the hypotenuse and an acute angle of one equal the hypotenuse and an acute angle of the other.

10. If two angles of a triangle are equal, the bisector of the third angle bisects the triangle.

11. If in $\triangle ABC$, $\angle B = \angle C$, then $AB = AC$.

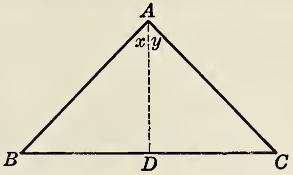
12. In congruent triangles, corresponding altitudes are equal.

13. If two sides of a triangle are equal, the bisector of their included angle cuts the figure into two congruent triangles.

14. Perpendiculars to the sides of an angle from a point on its bisector are equal.

THE ISOSCELES TRIANGLE

The base angles of an isosceles triangle are equal.



Given: $AB = AC$.

To prove: $\angle B = \angle C$.

Proof:

STATEMENTS	REASONS
Let AD bisect $\angle A$.	
1. In $\triangle$s ABD and ACD, $AD = AD$.	1. A line equals itself.
2. $AB = AC$.	2. Given.
3. $\angle x = \angle y$.	3. The bisector makes two equal $\angle$s.
4. $\triangle ABD \cong \triangle ACD$.	4. Two triangles are congruent if 2 sides and the included angle of one equal, etc.
5. $\angle B = \angle C$.	5. Corr. $\angle$s of $\cong$ $\triangle$s are =.

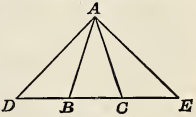
Exercises

1. An equilateral triangle is equiangular.

2. In an isosceles triangle, the exterior angles made by producing the base are equal.

3. Two isosceles triangles are congruent if the base and a base angle of one equal the base and a base angle of the other.

4. The lines from the vertex of an isosceles triangle to the trisection points of the base are equal. (To trisect is to divide into three equal parts.)



Exs. 5, 10

5. If the base BC of isosceles triangle ABC is extended so that $BD = CE$, then $\angle D = \angle E$.

If two angles of a triangle are equal, the sides opposite them are equal.

Given: $\angle B = \angle C$.

To prove: $AC = AB$.

Proof:

STATEMENTS	REASONS
Let AD bisect $\angle A$.	
1. In △s ABD and ACD, $AD = AD$.	1. A line equals itself.
2. $\angle x = \angle y$.	2. The bisector makes two equal ∠s.
3. $\angle B = \angle C$.	3. Given.
4. $\angle ADB = \angle ADC$.	4. If equals are subtracted from equals, etc.
5. $\triangle ADB \cong \triangle ADC$.	5. 2 △s are $\cong$ if 2 ∠s and the included side of one, etc.
6. $AB = AC$.	6. Corr. sides of $\cong$ △s are =.

Exercises

6. An equiangular triangle is equilateral.

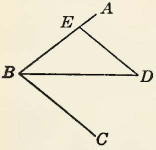
7. The legs of a right triangle are equal if one of its acute angles equals 45°.

8. If two angles of a triangle are 70° and 40°, the triangle is isosceles.

9. If any angle of an isosceles triangle is 60°, the triangle is equilateral. Prove two cases.

10. If $AB = AC$ and $\angle DAB = \angle EAC$, then $BD = CE$. (See figure for Ex. 5, p. 419.)

11. Construct an angle of 60°, 30°, 120°.



12. Trisect a right angle.

13. Find the number of degrees in each angle of an isosceles right triangle.

14. If BD bisects $\angle ABC$ and $EB = ED$, then $ED \parallel BC$.

15. If BD bisects $\angle ABC$ and $ED \parallel BC$, then $\triangle EBD$ is isosceles.

16. If $ED \parallel BC$ and $EB = ED$, then BD bisects $\angle ABC$.

17. If PQ is parallel to a leg KL of isosceles triangle KLM, then $\triangle PQM$ is isosceles.

SIMILAR TRIANGLES

Facts you already know. 1. In two similar triangles, the corresponding angles are equal, and the corresponding sides are proportional.

2. Two triangles are similar if two angles of one equal two angles of the other.

3. Two right triangles are similar if an acute angle of one equals an acute angle of the other.

4. In a proportion, the product of the means equals the product of the extremes.

Exercises

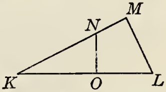
1. Are triangles KLM and PQR similar if:

(*a*) $\angle K = 41°$, $\angle L = 73°$, $\angle Q = 73°$, and $\angle R = 86°$?

(*b*) $\angle K = 67°$, $\angle L = 54°$, $\angle Q = 54°$, and $\angle R = 59°$?

2. A tower casts a shadow 200 ft. long when a vertical 8-ft. pole casts a shadow 10 ft. long. How high is the tower?

3. Triangles similar to the same triangle are similar to each other.



4. In $\triangle KLM$, NO is perpendicular to KL, and M is a right angle. Prove that $\triangle KNO$ is similar to $\triangle KLM$.

5. Two isosceles triangles are similar if a base angle of one equals a base angle of the other.

6. Two isosceles triangles are similar if the vertex angle of one equals the vertex angle of the other.

7. A line bisecting one side of a triangle and parallel to a second side bisects the third side and equals $\frac{1}{2}$ the second side.

8. For each of these proportions, read the triangles that must be proved similar, without attempting to draw a figure:

(*a*) $\frac{AG}{GC} = \frac{AD}{GF}$ (*b*) $\frac{PM}{PQ} = \frac{PR}{PS}$ (*c*) $\frac{HM}{HK} = \frac{KH}{LH}$

An *altitude* of a triangle is a perpendicular from a vertex to the opposite side.

9. In the acute triangle ABC, the altitudes BD and CE are drawn. Then $\frac{BD}{CE} = \frac{AB}{AC}$.

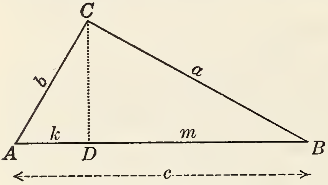
10. If in the same figure, the altitudes BD and CE cross at F, then $\frac{BF}{CF} = \frac{EF}{DF}$.

11. In similar triangles bisectors of corresponding angles have the same ratio as a pair of corresponding sides.

12. A line parallel to one side of a triangle divides the other two sides proportionally; that is, a side is to a segment of it as the other side is to its corresponding segment.

13. $\triangle ABC$ is a right triangle. CD is the altitude on the hypotenuse. Prove that:

(*a*) $\triangle ACD$ is similar to $\triangle ABC$



(*b*) $\triangle CBD$ is similar to $\triangle ABC$

(*c*) $\triangle ACD$ is similar to $\triangle CBD$

(*d*) $\frac{k}{b} = \frac{b}{c}$

(*e*) $\frac{m}{a} = \frac{a}{c}$

(*f*) From the result of (*d*), what does b^2 equal?

(*g*) From the result of (*e*), what does a^2 equal?

(*h*) Using the results of (*f*) and (*g*), find the value of $a^2 + b^2$.

In a right triangle, the square of the hypotenuse equals the sum of the squares of the legs.

Review Exercises

1. Find the hypotenuse of a right triangle whose legs are 3 and 4.

2. Find the diagonal of a rectangle the sides of which are 21 and 28.

3. Find the diagonal of a square whose side is 8.

4. Find the other leg of a right triangle if the hypotenuse and one leg are 26 and 24.

5. The base of a rectangle is 12, and the diagonal is 15. Find its altitude.

6. Find the altitude of an isosceles triangle whose base is 10 and whose legs are each 13.

7. Find the altitude of an equilateral triangle whose side is 20.

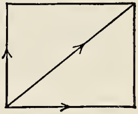
8. The bottom of a ladder 17 ft. long is 15 ft. from a wall. How high up the wall is the top of the ladder?

9. The perimeter of a rectangle is 28 ft., and its diagonal is 10 ft. Find the sides of the rectangle.

10. Find the side of a square whose diagonal is 10.

11. To find a distance AB, I measure AC 102 ft. and CB 136 ft. at right angles with each other. What is the length of AB?

12. A boat travels 40 ft. across a river while the current carries it 30 ft. down-stream. Draw the path of the boat, and find the distance it has moved.



13. A man walks at the rate of 5 mi. an hr. across the deck of a boat that is traveling 12 mi. an hr. What is his actual speed?

14. A force of 60 lbs. is pulling directly north, and a force of 80 lbs. directly east. If the resultant force is represented by the diagonal of the rectangle whose sides are proportional to the given forces, find its amount.

15. Emily is swimming 1 mi. an hr. across a river that flows 3 mi. an hr. At what speed is she moving?

THE INDIRECT PROOF

Pete Hill's father forbade Pete to use the new car except for doing errands for his mother. Last night Mr. Hill accused Pete of disobeying him. Mrs. Hill interceded, "He went to town on an errand for me. I don't believe he went anywhere else." "Well," said Mr. Hill, "we had the tank filled last night, didn't we?" "Yes," admitted Mrs. Hill. "And you see it's about empty now," continued Mr. Hill. "Yes," answered Mrs. Hill. "If he had only gone to town, he wouldn't have used that much gas, would he?" "No." "Then he must have disobeyed me," said Mr. Hill. And Mrs. Hill had to admit that he was right.

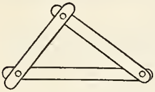
This type of argument is very common and very effective. It is called the indirect proof. Instead of starting out to prove your statement directly, you can suppose the opposite to be true and then show that this contradicts known facts. If there are more than two possibilities, you must show that all but the one you wish to establish contradict known facts. Let us apply it to geometry.

Illustration 1. The sides of a triangle are 3, 4, and 6. Show that it is not a right triangle.

Suppose the triangle is a right triangle.

1. Then $3^2 + 4^2$ would equal 6^2.	1. The sum of the squares of the legs of a right triangle equal, etc.
2. But this is impossible.	2. $9 + 16 = 25$ and not 36.
3. Therefore, the triangle is not a right triangle.	3. The supposition that it is a right triangle contradicts known facts.

Illustration 2. This triangle is made of three pieces of wood pinned together at the corners. Prove that it is rigid, that is, that it cannot change its shape.



Suppose it could change its shape. Then we would have different shaped triangles with the same three sides.

But this is impossible, for two triangles are congruent if the three sides of one equal the three sides of the other.

Therefore the triangle is rigid, for it is impossible that it could change its shape.

Exercises

Use the indirect method. Assume that the conclusion is not true, and then show that this leads to an impossible result:

1. If two angles of a triangle are unequal, the sides opposite them are unequal.

2. If any angle of one triangle is not equal to some angle of another triangle, the triangles are not congruent.

3. If a triangle is not isosceles, the bisector of an angle is not perpendicular to the opposite side.

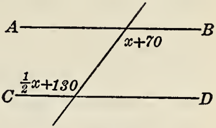
4. Two sides of one triangle equal two sides of another triangle. If their included angles are not equal, their third sides are unequal.

5. A triangle cannot have more than one right angle.

6. Two lines perpendicular to the same line are parallel. Suppose they meet, and consider the sum of the angles of the triangle formed.

7. The point (2, 3) is not on the line $2x + 3y = 12$.

8. The lines $2x + 3y = 5$ and $6x + 9y = 10$ are parallel. Suppose they are not parallel, and solve for their point of intersection.



9. *AB* and *CD* cannot be parallel. Suppose they are parallel, and solve for x and the angles.

10. A triangle cannot have two obtuse angles.

11. The sides of a triangle are 6, 8, and 11. Is it a right triangle? Prove your conclusion. What fact do you know about the sides of a right triangle?

12. Is it possible to construct an equilateral right triangle? Prove your conclusion.

The Indirect Proof in Life Situations

Use the indirect proof to convince the person that he is wrong.

Illustration. A customer returned the radio you sold him, saying that the loud speaker rattles. You try a new detector tube and it works quietly. Convince him that it was not the fault of the loud speaker.

Proof: "If it were the fault of the loud speaker, changing the detector tube would not correct it, would it?" "No." "But you see it does correct it, don't you?" "Yes." "Then it is not the fault of the loud speaker."

1. When the steam was on last night, water leaked from the radiator on the floor. The janitor says that the radiator is all right. Convince him that he is wrong.

2. A very heavy package was stolen from John's room. From the time it was seen there until it was missed, it is known that only Peter and the crippled Henry had been in there. You are the prosecuting attorney. Convince the jury that Peter is guilty.

3. The clock you sold yesterday has stopped. The customer returns it, saying that her maid wound it last night, but it will not run. You find that the spring is entirely unwound, so you wind it and it runs. The spring would be unwound if the clock had run down, or if the spring had broken. Convince the lady that her maid forgot to wind the clock.

4. You are the doctor. The child has a fever, rash, sore throat, and his tongue is red. He was vaccinated last year.

In measles there is fever, rash, tongue coated white, no sore throat. In scarlet fever there is fever, rash, tongue red, and sore throat. In smallpox there is fever and rash, but the disease is prevented by vaccination. In other diseases there is no rash. By the indirect method, diagnose the case.

Chapter 18

GENERAL REVIEW

Statistical Graphs

1. Make a bar graph of these statistics: The number of thousand miles of railway in the various countries is as follows: Canada, 42; France, 27; Germany, 33; Great Britain, 20; India, 42; Italy, 10; Japan, 13; Russia, 48; and U.S., 248.

2. Make a smooth-curve graph: The size of the American merchant marine in millions of tons was: 1830, 1; 1850, 3.5; 1870, 4; 1890, 4.4; 1910, 7.5; and 1930, 16.

3. Make a broken-line graph: The number of deaths in New York City from alcohol was: 1914, 660; 1915, 562; 1916, 687; 1917, 560; 1918, 252; 1919, 176; 1920, 98; and 1921, 119.

4. Make a circle graph: The number of million acres of irrigated land is: No. America, 27; So. America, 7; Europe, 15; Asia, 141; and Africa, 10.

5. Find the average, median, and mode of these test marks: 5, 6, 6, 6, 6, 7, 7, 8, 8, 9, 10, 10.

6. Make a frequency polygon and a normal frequency curve: In a certain school the height in inches of 9-yr.-old children was:

Height in inches	46	47	48	49	50	51	52	53	54
Number of pupils	3	10	42	112	143	114	47	9	2

What height was the mode?

Formulas

7. In $f = a + (n - 1)d$, find f when $a = 7$, $n = 20$, and $d = 3$.

8. In $K = \frac{1}{2} mv^2$, find K when $m = 5000$ and $v = 12$.

9. In $s = \frac{I}{ma}$, find s when $I = 380$. $m = 19$. and $a = 12$.

10. In $t = 2\pi\sqrt{\frac{s}{g}}$, find t when $s = 98$ and $g = 32$.

11. In $A = \sqrt{s(s-a)(s-b)(s-c)}$, find A when $a = 12$, $b = 9$, and $c = 11$, if $s = \frac{1}{2}(a + b + c)$.

12. In a certain state, fire-insurance companies are required to pay partial losses according to the formula $p = \frac{il}{.8\,v}$, where p is the amount they must pay, i the amount of insurance on the house, l the amount of the loss, and v the value of the house. If a house valued at $12,000 is insured for $7200, what should the company pay in case of a fire loss of $2800?

13. A merchant sold a suit for $47.20 making a profit of 18% of the cost. What was the cost. Formula: $c = \frac{s}{1+p}$

14. Make a graph of the formula $s = 5t + 12$ from $t = 0$ to $t = 5$.

Geometric Measurement

15. Find an angle that is 30° more than four times its supplement.

16. How many degrees are there in the angle made by the hands of a clock at 1:15 P.M.?

17. When an angle grows larger, what change takes place in its supplement? In its vertical angle? Are supplementary angles ever equal? Explain.

18. The sum of the angles of a polygon (a figure having any number of straight sides) is $s = (n - 2)180$, where n is the number of sides. Find the sum of the angles of a polygon having 10 sides, having 12 sides.

19. Draw an angle, and construct an angle $3\frac{1}{2}$ times as large.

20. Construct an angle of $22\frac{1}{2}°$, one of 135°.

21. Find the square root of 3847 to hundredths.

22. Could you find the square root of 384.7 by simply pointing off one place in the answer to Exercise 21? Explain.

23. Find the length of a root 3 rectangle whose width is 10 in.

24. Express in a short form (*a*) $xxxxxx$, (*b*) $y + y + y + y$.

25. Find to four significant figures the value of $\sqrt{6783}\ \sqrt{6783}$.

Algebra as a Language

Write in the shorthand of algebra:

26. The sum of two numbers, the difference of two numbers, the difference of the squares of two numbers, the sum of the cubes of two numbers.

27. Three times a number decreased by 13

28. Five times the sum of two numbers increased by 5

29. Translate these into English, referring to x as a number, etc.:

(*a*) $2x - 7$ (*b*) $x^2 + 5$ (*c*) $x^2 + y^2$ (*d*) $\sqrt{x + y}$

Algebraic Numbers

30. What is meant by: A temperature of $-20°$? A profit of $-\$24$? A height above sea level of -240 ft.? Emily will be 11 yrs. old in -4 yrs.?

31. Find the value of: 18 decreased by -5, the sum of -7 and 3.

32. Add:

$$(a)\ \begin{array}{r} 5x \\ -2x \\ -7x \\ x \\ \hline \end{array} \qquad (b)\ \begin{array}{r} -3m^2n \\ -8m^2n \\ 9m^2n \\ 2m^2n \\ \hline \end{array} \qquad (c)\ \begin{array}{r} 3ac \\ -4ac \\ ac \\ -5ac \\ \hline \end{array} \qquad (d)\ \begin{array}{r} -7xyz \\ 4xyz \\ 5xyz \\ -2xyz \\ \hline \end{array}$$

33. If I start at -7 and go 7 to the right, at what point do I arrive?

34. Virginia earns d dollars a week for 7 weeks and spends c¢ a day during that time. How many cents does she save?

Subtraction

35. By how much does (*a*) 8 exceed 5? (*b*) a exceed 7? (*c*) 10 exceed x? (*d*) m exceed n? (*e*) -3 exceed -7? (*f*) 8 exceed -2? (*g*) -3 exceed $+5$?

36. Caius was born in -18 and died in $+46$. How many years did he live?

37. Remove parentheses, and collect terms:

(*a*) $7a - (3a - 4) + 2a$

(*b*) $4x - 2 + (3 - x) - 1$

(*c*) $-(7-5n)-(-3n+4)$
(*d*) $2+9c-(3-c)+(1-c)$

38. How many sharps are there in the key of E? Of A♭? Of F? Of B?

Multiplication

39. Multiply:

(*a*) $-7(+5)$
(*b*) $2a(-3a^3)$
(*c*) $-5m^5(2m^2)$
(*d*) $(-1)^3$
(*e*) $-4x^3(-3x)(-2x^5)$
(*f*) $5a^3b^2cd^4(-3ab^3c^3d)(0)$
(*g*) $-k^2n^3(5k^3-2kn^2+3n^4)$
(*h*) $7abc(a^2b-5ac^3-3b^3c)$

40. Factor:

(*a*) $4ax-6bx^3$
(*b*) $9m^2r^3-3m^2r$
(*c*) $\frac{1}{2}\pi r^2+\frac{1}{2}\pi rh$
(*d*) $.3x^2-1.5xy$

Division

41. Divide:

(*a*) $x^5 \div x^3$
(*b*) $-7 \div 2$
(*c*) $-16a^5 \div 2a^3$
(*d*) $-18cx^2 \div (-6x)$
(*e*) $(8a^2b^3-6ab^4) \div (-2ab^2)$
(*f*) $(-9m^3x+21mx^2) \div (-3mx)$
(*g*) $(14abc-7ac) \div 7ac$
(*h*) $(x^{16}+x^{12}-x^8) \div x^8$

Equations

42. Solve and check:

(*a*) $3x-7=13-2x$
(*b*) $8y+11=y-3$
(*c*) $1.3m+2=m+5.6$
(*d*) $3(4-2x)=9-(2x+1)$
(*e*) $\frac{3r}{2}-\frac{2r}{5}=r+1$
(*f*) $\frac{x+2}{4}-\frac{2x-3}{3}=-1$
(*g*) $\frac{3}{x}-\frac{5}{x+4}=0$

Problems

43. Find two supplementary angles in the ratio 3:5.

44. One of two vertical angles is 20° more than twice a number, and the other is 40° less than 3 times that number. Find the number and the size of the angles.

45. If one of two vertical angles were 20° less than it is, and the other were 20° larger than it is, they would be in the ratio $\frac{3}{7}$. Find the angles.

46. Find the number of degrees in the angles of a triangle if they are in the ratio 4:5:6.

47. One angle of a triangle is twice another, and the third is 30° less than their sum. Find the three angles.

48. A broker makes $\frac{1}{4}\%$ on sales. How much must he sell to make a yearly salary of $3000?

49. The expenses of a city are $32,000, and the city's assessed value is $2,048,000. Express the tax rate as a decimal, as a per cent.

50. Mr. Smith has two investments totaling $30,000 that furnish him an income of $1320 a year. If the rates are 5% and 4%, how much did he invest at each rate?

51. Mr. Washburn invested $5000 at 4%. How much must he invest at 7% to make his income 5% of his total investment?

52. Fred leaves a town at 6 A.M. traveling 30 mi. an hr. At 9 A.M. George follows at 45 mi. an hr. In how many hours will George overtake Fred?

53. In the last exercise, if George had started from another town 40 mi. from the point from which Fred started and through which Fred passed on his way, in how many hours would George overtake Fred?

54. Two cars leave a town going in opposite directions at rates that differ by 5 mi. an hr. What are their rates if they are 330 mi. apart at the end of 6 hrs.?

55. To pass a period of 3 hrs. which he must spend in a strange town, Robert takes a ride out into the country at the rate of 20 mi. an hr. and walks back at 4 mi. an hr. How far can he go?

56. How many pounds of peanuts worth 20¢ a lb. should the H and K Company mix with 60 lbs. of Brazil nuts worth 50¢ a lb. to make a mixture worth 32¢ a lb.?

57. An alloy contains 60% lead and 40% tin. How much tin should be added to 100 lbs. of it to make an alloy 40% lead?

58. Charles has a batting average of .280 for the first 25 times at bat. How many hits must he make in succession to raise his batting average to .400?

59. One of two alternate interior angles of parallel lines is represented by $4n + 30$ and the other by $2n + 60$. What is the value of n?

60. Find each angle of an isosceles triangle if each of the two base angles is 15° larger than the vertex angle.

Sets of Equations

61. Make a graph of each of these equations:

(*a*) $2x + 3y = 12$ (*c*) $y = 3x - 4$

(*b*) $x - 4y = 8$ (*d*) $x = 6$

62. Solve graphically:

(*a*) $2x + 3y = 12$, $x - y = 1$ (*b*) $3x - 2y = 12$, $2x + 5y = 8$

63. Solve algebraically:

(*a*) $3x + y = 17$, $2x - y = 8$ (*c*) $3x - 2y = 3$, $x + 4y = 29$

(*b*) $7x + 5y = 23$, $x + y = 3$ (*d*) $7x + 2y = 34$, $2x + 3y = 17$

64. At what point does $5x - 2y = 10$ cut the x axis? The y axis?

65. Is the equation $3x - y = 7$ satisfied by the point (2, 1)? By (– 2, 1)? By (2, – 1)?

66. Solve for x and y:

(*a*) $\frac{x}{2} - \frac{y}{5} = 1$, $\frac{3x}{4} + \frac{y}{5} = 4$ (*d*) $x - 4y = 7$, $\frac{x+5}{4} + \frac{y-2}{3} = 1$

(*b*) $.4x + .05y = 5$, $.03x + .15y = 3.3$ (*e*) $.2x + .01y = .8$, $\frac{x+7}{2} - \frac{y-2}{3} = -1$

(*c*) $\frac{x}{2} + \frac{y}{3} = 2$, $x - y = 1$ (*f*) $\frac{3}{x} + \frac{5}{y} = \frac{11}{4}$, $y = 2x$

Problems in Two Letters

67. If the length of a rectangle is increased 2 and its width is decreased 1, its area remains the same, but if its length is increased 4 and its width decreased 3, its area is decreased 20. Find its length and width.

68. Mr. Banker loaned \$4000 at one rate and \$5000 at another rate. He received from them \$480 a yr. interest. If he had loaned the \$4000 at the second rate and the \$5000 at the first rate, his yearly income would have been \$465. What were the two rates?

69. \$8000 is invested, part at 6% and part at 5%. How much is invested at each rate (*a*) If the income from the first part is twice that from the second part? (*b*) If the total income is \$470? (*c*) If the income from the first part exceeds that from the second part by \$40?

70. How many pounds of tea at 35¢ a lb. and how many at 50¢ a lb. must be taken to make a mixture of 200 lbs. worth 38¢ a lb.?

71. In a race of 100 yds. Chester beat Tom by $2\frac{1}{2}$ secs. In a second race Tom was given a start of 4 yds., but was still beaten by 2 secs. What were their rates?

72. The sum of the digits of a two-digit number is 11. If the digits were reversed, the number would be increased by 27. What is the number?

Formulas

73. Solve $l = \frac{I}{ma}$ for m. Then find m when $l = 30$, $I = 600$, and $a = 8$.

74. The temperature, volume, and pressure of a gas are related according to the formula $P = \frac{C\left(1 + \frac{T}{273}\right)}{V}$. Solve for T. Then find T when $C = 12$, $V = 2$, and $P = 8$.

75. The formula $\frac{I}{B} = \frac{R - D_i}{D_b - R}$ is used to find the size of the image I of a body B, made by a lens. Solve for R. Find the radius if $I = 3$, $B = 5$, $D = 7$, and $D_b = 15$. Check by substituting in the original equation.

General Solution

76. (*a*) A man invests d dollars, part at rate p and part at rate r. If his income is i dollars a year, how much has he at each rate?

(*b*) Using the solution to (*a*) as a formula, solve: A man invests \$800, part at 4% and part at 5%. If his income is \$36 a yr., how much has he at each rate?

77. (*a*) A merchant has p lbs. of candy worth c¢ a lb. How much candy worth a¢ a lb. must he use with it to make a mixture worth m¢ a lb.?

(*b*) Using the solution to (*a*) as a formula, solve: A merchant has 300 lbs. of candy worth 60¢ a lb. How much candy worth 30¢ a lb. must he use to make a mixture worth 40¢ a lb.?

78. (*a*) A train leaves a station and travels m mi. an hr. After h hrs. a faster train follows at r mi. an hr. In how many hours will the second train overtake the first?

(*b*) Using the solution to (*a*) as a formula, solve: A train leaves a station and travels 25 mi. an hr. After 4 hrs. a second train follows at 35 mi. an hr. In how many hours will the second train overtake the first?

Business Problems

79. A merchant sells shoes that cost \$3.50 a pair for \$4.90. His profit is what per cent of the cost? Of the selling price?

80. A dealer buys ladies' hats at \$3.60 each. At what price must he sell them to make 25% of the cost? 25% of the selling price?

81. Find the bank discount and the proceeds on:

(*a*) \$200 at 6% for 48 d.
(*c*) \$180 at 5% for 50 d.
(*b*) \$350 at 6% for 3 mo.
(*d*) \$480 at $4\frac{1}{2}$% for 72 d.

82. Using the compound-interest table, find the amount of:

(*a*) \$500 at 6% compounded annually for 10 yrs.
(*b*) \$300 at 4% compounded semi-annually for $4\frac{1}{2}$ yrs.

83. Using logarithms, find the amount of:

(*a*) \$200 at 4% compounded quarterly for 20 yrs.
(*b*) \$600 at 6% compounded semi-annually for 80 yrs.
(*c*) \$1095 at 6% compounded semi-annually for 25 yrs.
(*d*) \$24 at 6% compounded semi-annually for 350 yrs.

Ratio and Proportion

84. Solve for x:

(*a*) $\frac{x}{3} = \frac{5}{9}$ (*c*) $\frac{x-1}{4} = \frac{3}{8}$ (*e*) $\frac{x-3}{x+2} = \frac{2x-5}{2x+6}$

(*b*) $\frac{5}{2} = \frac{7}{x}$ (*d*) $\frac{7}{13} = \frac{x+2}{2x+3}$ (*f*) $\frac{x+2}{x-5} = \frac{x-4}{x-5\frac{3}{4}}$

85. (*a*) Find two numbers in the ratio $a:b$ whose sum is s.

(*b*) Using the answer to (*a*) as a formula, find two numbers in the ratio 5:8 whose sum is 78.

86. Three partners invested $6000, $7000, and $8000 in an enterprise. If their profits are to be divided in the ratio of their investments, how should a profit of $5292 be divided?

87. When coal burns, a gas called carbon dioxide ($C + O\,2$) is formed. Assuming that the coal is 80% carbon, how many tons of carbon dioxide are formed by the burning of 15 T. of coal? The atomic weights are C = 12 and O = 16.

88. What is the width of a root 3 rectangle whose length is 40 in.?

89. A low-pitched scale has vibration rates: C = 24, D = 27, E = 30, F = 32, G = 36, A = 40, B = 45, and C = 48.

(*a*) Write the vibration rates of a scale in which C = 120.

(*b*) Which is the worst discord: C and D, or E and F?

(*c*) Do A, F, and C form a pleasing chord?

(*d*) What vibration rate should E have, to the nearest tenth, so that the ratio E:D would equal that of D:C?

90. If the amount of water carried by a pipe varies as the square of the diameter, a 4-in. pipe will carry how many times as much water as a 2-in. pipe? As a 1-in. pipe?

91. How high is a tree that casts a shadow 27 ft. long when an 8-ft. vertical pole casts one 3 ft. long?

92. A line parallel to a side of a triangle cuts a second side into segments of 7 and 5. If the length of the third side is 18, what are the lengths of the segments into which the line cuts it?

93. On a map of Ethiopia, the distance from Adowa to the Eritrean border is 4 cm. and from Adowa to Addis Ababa is 6.9 cm. If Adowa is 345 mi. from Addis Ababa, how far is Adowa from the border.

94. Make a formula to fit these tables:

(*a*)

t	1	2	3	4	5
d	8	16	24	32	40

(*b*)

A	15000	16000	17000
C	8000	9000	10000

(*c*)

w	1	2	3	4	5
f	7	11	15	19	23

(*d*)

P	1	2	3	4	5
V	60	30	20	15	12

95. Dorothy, who weighs 84 lbs., sits 6 ft. from the support of a teeter board. How far from the support should Sarah, who weighs 72 lbs., sit?

The Straight Line

96. At what point will $3x + 2y = 8$ cut the y axis? The x axis?

97. Does $4x - y = 7$ pass through (*a*) The point (1, 1)? (*b*) The point (2, 1)? (*c*) The point (3, 5)?

98. Give the slope and y intercept of:

(*a*) $y = 2x + 3$ (*c*) $3x + y = 7$

(*b*) $y = x - 4$ (*d*) $y = 5$

99. Which of these lines is parallel to $y = 3x + 4$:

(*a*) $y = 2x + 4$ (*b*) $y = 3x - 2$ (*c*) $y = -3x - 4$

Scale Drawing

100. By drawing the figure carefully to scale and measuring the lines, solve these problems:

(*a*) In triangle ABC, $AB = 400$ yds., $\angle A = 52°$, and $\angle B = 64°$. Find the length of AC.

(*b*) In triangle ABC, $\angle C = 112°$, $AC = 200$ ft., and $BC = 300$ ft. What is the length of AB?

(*c*) To find the height of her school, Emily measured a distance of 60 ft. out from the school, and with her clinometer found the angle of elevation of the top to be 42°. How high was the school?

(*d*) A tennis court is 78 ft. long by 27 ft. wide. Draw a plan using as a scale 1 in. = 12 ft.

Trigonometry

101. How high is a flag-pole that casts a shadow 80 ft. long when the angle of elevation of the sun is 28°?

102. What is the angle of elevation of the sun when a vertical pole 60 in. long casts a shadow 54 in. long?

103. To find the "airman's ceiling," a beam of light is thrown vertically to the clouds from a search-light 600 ft. distant from the observer. If the angle of elevation of the spot of light on the cloud is 74°, what is the height of the "ceiling"?

104. To find the distance from the top of a cliff to the opposite bank of a river, Paul first dropped a stone. It struck the water in 3 sec. Then he found the height of the cliff by using the formula $d = 16\,t^2$. Next he measured the angle of depression of the opposite bank. This he found to be 10°. How wide was the river and how far was Paul from the opposite bank?

Exponents and Logarithms

105. Find the value of:

(*a*) 2^4 (*c*) $8^{\frac{1}{3}}$ (*e*) $32^{\frac{1}{5}}$ (*g*) $16^{\frac{5}{4}}$
(*b*) $16^{\frac{1}{2}}$ (*d*) $9^{\frac{3}{2}}$ (*f*) $8^{\frac{2}{3}}$ (*h*) $25^{\frac{3}{2}}$

106. Find to the nearest hundredth the value of:

(*a*) $6^{\frac{1}{2}}$ (*b*) $8^{\frac{1}{2}}$ (*c*) $2^{\frac{3}{2}}$ (*d*) $4^{\frac{1}{4}}$

107. Find the logarithm of:

(*a*) 7 (*d*) 3.8 (*g*) 69.4 (*j*) .0835
(*b*) 80 (*e*) 38 (*h*) 732 (*k*) 9.99
(*c*) 700 (*f*) .073 (*i*) .564 (*l*) 23.1

108. Find the number whose logarithm is:

(*a*) .9085 (*c*) 5.3181 (*e*) 2.6079
(*b*) 2.7803 (*d*) $.9800 - 2$ (*f*) $.1884 - 1$

109. Using logarithms, find:

(*a*) $\dfrac{89.2}{6.14}$ (*b*) $3690 \times .0512$ (*c*) $\sqrt[12]{2}$

110. Using the slide rule, find:

(*a*) $\dfrac{36.5}{14.8}$ (*b*) 2.96×78.5 (*c*) $\sqrt{183}$

Polynomials

Add and check:

111. $3A - 2B + 4C,\ 2A + B - 7C,\ -6A + 3B + C$
112. $-m + 2a - 5x,\ 3x - 4m - 5a,\ 4a + 3m + x$
113. $7x^2 - 3x - 12,\ 4x - 7 - x^2,\ 14 - 5x^2 - x$
114. $M - 4N,\ 3N - 5M,\ M + N,\ 2M - 3N$
115. $2x_1 - 3x_2 + x_3,\ 2x_2 - x_3 - x_1,\ -x_3 + 4x_1 - x_2$
116. $1.3a - 2.7b + 4c,\ 2.5a + 1.4b - c,\ a + b$
117. $k - .03m,\ m - .55k,\ .5k - m,\ 2m - 2.4k$
118. $\frac{1}{2}a - \frac{1}{3}b + \frac{1}{4}c,\ \frac{1}{2}b - \frac{1}{3}c + \frac{1}{4}a,\ \frac{1}{2}c - \frac{1}{3}a + \frac{1}{4}b$

Subtract and check:

119. Subtract $5a - 2b + c$ from $2a - 3b + c$
120. Subtract $-3X + 2Y - Z$ from $X - Y + 4Z$
121. From $2m - 3n + p$, take $5m + n - 3p$
122. From $7x_1 - 2x_2 - 3x_3$, take $x_1 + x_2 - 4x_3$
123. Take $3a + 5$ from $7 - 2a$
124. Take $-x + 2y - 5z$ from $4x + z$
125. From $3.7x - 2.9y - z$, take $4.7x + 3.1y - .4z$
126. Subtract $\frac{1}{2}x - \frac{1}{3}y + \frac{1}{4}z$ from $\frac{1}{3}x - \frac{1}{4}y + \frac{1}{2}z$

Multiply and check:

127. $x^2 + 2x - 3$ by $3x - 4$
128. $3a^3 - 2a^2 + 7a - 5$ by $2a - 3$
129. $(5x^2 - 3xy - 2y^2)(3x^2y - 4xy^2)$
130. $(4p^2 - 2q + 3r)(2p^2 + q - 3r)$
131. $x^4 + x^3 + x^2 + x + 1$ by $x - 1$
132. $M^2 - .2M + .04$ by $.1M^2 + 1.2M - .3$
133. $\frac{1}{2}a^2 - \frac{1}{3}a + \frac{1}{4}$ by $\frac{1}{3}a^2 - \frac{1}{4}a + \frac{1}{2}$
134. $(x^{n+1} - 3x^n - 2x^{n-1})(x - 2)$

Divide and check:

135. $3x^2 - 4x + 1$ by $x - 1$
136. $8x^3 - 27y^6$ by $2x - 3y^2$
137. $(6A^3 - 5A^2 - 4A - 3) \div (2A - 3)$

138. $(x^4 + x^2y^2 + y^4) \div (x^2 - xy + y^2)$

139. $\dfrac{a^4 + 11\,a^3 + 23\,a^2 - 55\,a - 140}{a^2 - 5}$

140. $\dfrac{M^3 - 2.1\,M^2 + 2.18\,M - 1.32}{M - 1.2}$

141. $(a^4 - .0016) \div (a - .2)$

142. $(x^{n+4} + 3\,x^{n+1} - 10\,x^{n-2}) \div (x^3 - 2)$

Factoring

Factor and check:

143. $a^2 - 25$

144. $9\,x^2 - 49$

145. $c^2 - \frac{4}{9}$

146. $a^4 - 16\,b^2$

147. $M^2 - .0025$

148. $y^2 - \dfrac{1}{z^2}$

149. $\dfrac{1}{x^2} - \dfrac{4}{y^2}$

150. $k^4 - 16\,n^4$

151. $h^3 - 36\,h$

152. $4\,D^2 - 36\,g^2$

153. $6\,t^2 - 5\,t - 1$

154. $2\,a_1^2 - 3\,a_1a_2 + a_2^2$

155. $3\,v^2 - 3\,v - 6$

156. $16\,m^2 + 1 - 8\,m$

157. $2\,x^3 - 3\,x^2 - 9\,x$

158. $-\,x^2 + 24\,x - 44$

159. $x^4 - 10\,x^2y^2 + 9\,y^4$

160. $a^5 - 13\,a^3 + 36\,a$

161. $x^{n+2} - x^{n+1} - 20\,x^n$

162. $x^4 - 4.01\,x^2 + .04$

Find the value of:

163. $17^2 - 16^2$

164. $344^2 - 342^2$

165. $783^2 - 217^2$

166. $443^2 - 433^2$

167. $451^2 - 449^2$

168. $501^3 - 501 \times 499^2$

Fractions

Reduce to lowest terms:

169. $\dfrac{15\,a^2b^3}{25\,a^3b}$

170. $\dfrac{-\,14\,MN^3}{21\,M^4N^2}$

171. $\dfrac{-\,22\,a^3b^5}{-\,33\,a^3b^2}$

172. $\dfrac{15\,Xx^2}{-\,5\,Xx}$

173. $\dfrac{8\,x^3y^2z}{8\,x^3y^2z}$

174. $\dfrac{a^2 - 9\,b^2}{a^2 - 3\,ab}$

175. $\dfrac{5\,x^2y}{10\,x^3y^2 + 15\,x^2y^3}$

176. $\dfrac{5 - 5\,y^2}{4\,y + 4}$

177. $\dfrac{6\,m^2 - 24\,n^2}{3(m - 2\,n)^2}$

178. $\dfrac{.3\,x^2 - 1.2\,y^2}{.01\,x + .02\,y}$

179. $\dfrac{x^2 - 16}{x^2 - 8\,x + 16}$

180. $\dfrac{3\,x^2 - 75}{2\,x^2 + 8\,x - 10}$

181. $\dfrac{7\,x^2 - 21\,x}{14\,x^2 - 14\,x - 84}$

182. $\dfrac{2\,a^2 - 5\,a - 3}{a^2 + 4\,a - 21}$

183. $\dfrac{3\,m^2 - 9\,m - 12}{3\,m^2 + 15\,m + 12}$

184. $\dfrac{x^2 - .1\,x - 1.1}{x^2 - 1.3\,x + .22}$

Multiply:

185. $\dfrac{72\,x}{49\,x^3} \cdot \dfrac{35\,x^4}{9\,x^2} \cdot \dfrac{7\,x^5}{8\,x}$

186. $5 \cdot \dfrac{3\,b^5}{a^2} \cdot \dfrac{4\,a^3}{25\,b}$

187. $\left(\dfrac{3\,m}{2\,p}\right)^3 \dfrac{4\,p^4}{6\,n^2} \left(\dfrac{2\,n}{3\,p^2}\right)^2$

188. $\dfrac{4\,\pi r^2}{6\,e} \left(\dfrac{3\,e}{2\,\pi r}\right)^3 \dfrac{4\,\pi r^3}{3}$

189. $\dfrac{-5\,a^2x^3}{3\,m^2y} \cdot \dfrac{6\,m^3y}{-10\,ax^3}$

190. $\dfrac{2\,k^2 + 6}{3\,k^5} \cdot \dfrac{21\,k^3}{7\,k^2 + 21}$

191. $\dfrac{(3\,x + 2\,y)^2}{3\,x - 2\,y} \cdot \dfrac{2\,x}{6\,x^2 + 4\,xy}$

192. $\dfrac{S^2 - 4}{S^2 - 9} \cdot \dfrac{3\,S + 9}{6\,S - 12}$

193. $\dfrac{3\,x^2 + 9\,x - 12}{2\,x^2 + 8\,x - 10} \cdot 2\,x^2$

194. $\dfrac{i^2 - i - 6}{i^2 + 3\,i + 2} \cdot \dfrac{i^2 - 1}{i^2 - 4\,i + 3}$

Divide:

195. $\dfrac{8\,A^3B}{5\,AC^2} \div \dfrac{4\,A^2B}{10\,C^3}$

196. $\dfrac{3\,M^2}{5\,S} \div 2\,MS$

197. $7\,x^3y \div \dfrac{7\,x^2}{2\,y}$

198. $\dfrac{-10\,a^2b}{3\,x^3} \div \dfrac{6\,ab^2}{9\,bx^4}$

199. $\dfrac{-5\,x}{4} \div \dfrac{3}{-4}$

200. $\dfrac{5\,H^2}{3\,H^2 - 3} \div \dfrac{5\,H^2}{3\,H^2 - 3\,H}$

201. $\dfrac{a^2 - b^2}{a^2 + b^2} \div \dfrac{a - b}{2\,a^2 + 2\,b^2}$

202. $\dfrac{(x - y)^2}{2\,x + y} \div \dfrac{2\,x - 2\,y}{6\,x^2 + 3\,xy}$

203. $\dfrac{a^2 - b^2}{x^2 - y^2} \div \dfrac{b - a}{y - x}$

204. $\dfrac{ax - 2\,a}{4 - x^2} \div \dfrac{-a}{(x + 2)}$

Add:

205. $\dfrac{5\,m}{3} - \dfrac{m}{6} + \dfrac{3\,m}{2}$

206. $\dfrac{2\,a^2B}{10} - \dfrac{3\,a^2B}{5} + \dfrac{a^2B}{2}$

207. $\frac{5}{3x} + \frac{2}{x} - \frac{1}{5x}$

208. $\frac{5a}{2} - 3a + \frac{a}{3}$

209. $\frac{2}{3x^2} + \frac{5}{6x} - \frac{1}{x^3}$

210. $\frac{a+b}{2} - \frac{a-b}{3}$

211. $\frac{x+y}{2xy} - \frac{2}{y}$

212. $\frac{3}{p-q} - \frac{2}{q-p}$

213. $\frac{4V^3 - 2V^2 + 1}{3V^2} - \frac{V^2 - 3}{2V}$

214. $\frac{3}{1-m} + \frac{5m}{m^2 - 1}$

Quadratic Equations

Solve and check:

215. $x^2 + x - 12 = 0$
216. $N^2 - N - 2 = 0$
217. $r^2 + 3r = 10$
218. $f^2 - 21 = 4f$
219. $x + 20 = x^2$
220. $2j^2 + 3j - 2 = 0$
221. $12z^2 + 5z - 2 = 0$
222. $12h^2 + 17h + 6 = 0$
223. $6e^2 - 23e + 20 = 0$
224. $10x^2 + .7x - .03 = 0$

Solve, leaving the answer in radical form:

225. $3R^2 - 2R - 6 = 0$
226. $2p^2 - 4 = 3p$
227. $2c^2 + 5c = 4$
228. $5I^2 - 8I + 2 = 0$
229. $b^2 - 1.2b + .1 = 0$
230. $.02y^2 = .7y + .4$

Find the roots to the nearest tenth:

231. $3x^2 - 4x - 2 = 0$
232. $2x^2 + 8x + 3 = 0$
233. $x^2 - 5x + 3 = 0$
234. $4x^2 - 9x + 3 = 0$

Solve and group your answers:

235. $x + y = 7$
$x^2 - xy = 15$
236. $2k - m = 5$
$k^2 + m^2 = 10$
237. $a + 3d = 7$
$d^2 + ad = 3$
238. $r = 2v$
$2r^2 - 3rv = 18$
239. $2w + 5z = 7$
$2w^2 + 5z^2 = 7$
240. $\frac{1}{2}R - \frac{2}{3}r = 0$
$.5R^2 + .01Rr = 8.12$
241. $n_1^2 + n_1n_2 + n_2^2 = 3$
$n_1 + n_2 + 1 = 0$
242. $H^2 - J^2 = 5$
$H + J = 5$

Radicals

Simplify:

243. $5\sqrt{\frac{1}{5}}$

244. $2\sqrt{\frac{1}{8}}$

245. $3\sqrt{96}$

246. $2\sqrt{80}$

247. $\dfrac{\sqrt{12}}{\sqrt{3}}$

248. $\dfrac{2\sqrt{5}}{5\sqrt{2}}$

249. $\dfrac{4}{\sqrt{6}}$

250. $\dfrac{3\sqrt{15}}{5\sqrt{12}}$

251. $6\sqrt{\frac{3}{8}}$

252. $9\sqrt{\frac{4}{27}}$

253. $5\sqrt{.08}$

254. $5\sqrt{3\frac{1}{5}}$

255. $a^3\sqrt{\dfrac{b}{a^3}}$

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